MATH 115A
MIDTERM EXAMINATION
SOLUTIONS

February 12, 2003

Please show your work. Except on True/False problems, you will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. You are not allowed to consult notes, books, friends, nor use computational devices of any kind. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see your professor *within 15 calendar days of the examination.*

Name:Solutions   Sid:000000000

#1	#2	#3	#4	#5		Total

Problem 1. Let $P_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3 with real coefficients. Let

$$W = \{f \in P_3(\mathbb{R}) : f(0) = 2f'(0)\}.$$

(a) Show that W is a subspace of $P_3(\mathbb{R})$.
(b) Find a basis of W.
(c) Find the dimension of W.

Solution. (a, first way) We must show that W is closed under addition and scalar multiplication. Suppose that $f, g \in W$ and $\alpha \in \mathbb{R}$. Then by the definition of W, $f(0) = 2f'(0)$ and $g(0) = 2g'(0)$. Now,

$$(f+g)(0) = f(0) + g(0) = 2f'(0) + 2g'(0) = 2(f' + g')(0) = 2(f+g)'(0),$$

so that $f + g \in W$. Similarly,

$$(\alpha f)(0) = \alpha f(0) = \alpha 2 f'(0) = 2(\alpha f)'(0),$$

so $\alpha f \in W$.
(b&c, first way) Note that W is a subspace of a three-dimensional space. Thus there are 4 possibilities for the dimension of W: $0, 1, 2, 3$ and 4. Clearly, not every polynomial is in W (e.g., the constant one is not), so $W \neq P_3(\mathbb{R})$. Also, note that x^2, x^3 and $x+2$ are all in W. These polynomials form a linearly independent set. Indeed, assume that $\alpha(x+2) + \beta x^2 + \gamma x^3 = 0$; thus this equality holds for all x. Evaluating at $x = 0$ gives $\alpha = 0$; dividing by x gives $\beta + \gamma x = 0$ for all $x \neq 0$, so that $\beta = \gamma = 0$. Thus $\{x+2, x^2, x^3\}$ is a linearly independent set of three elements of W. Thus $\dim W \geq 3$; since we saw before that $\dim W < 4$, it follows that $\dim W = 3$. Furthermore, it follows that $\{x+2, x^2, x^3\}$ is a basis for W.
(a&b&c, second way) Define $T : P_3(\mathbb{R}^3) \to \mathbb{R}$ by

$$T(f) = f(0) - 2f'(0).$$

Since the maps $f \mapsto f(0)$ and $f \mapsto f'(0)$ are clearly linear, it follows that T is a linear transformation.
Note that

$$T(f) = 0 \Leftrightarrow f(0) = 2f'(0).$$

Thus by definition,

$$W = N(T).$$

It follows that W is a subspace. Since $T(x) = -2 \neq 0$, we get that the range of T is a nonzero subspace of $\mathbb{R}$, and thus T is onto. By the rank-nullity theorem, it follows that $\dim W = \dim N(T) = \dim(P_3(\mathbb{R})) - \dim(\mathbb{R}) = 4 - 1 = 3$. Thus any basis of W must have three elements. As before, note that $x+2, x^2, x^3$ are linearly independent and belong to W and so form a basis of W.

Problem 2. True or False. For each of the following statements, indicate if it is true or false. This problem will be graded as follows: you will receive 5 points for a correct answer, 0 points if there is no answer, and -5 points if the answer is wrong.

1. If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
2. If S is a subset of a vector space V, then $\text{span}(S)$ equals the intersection of all subspaces of V that contain S.
3. If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

Solution. (1) is false, since the vector space structure of W need not agree with that of the ambient space. For example, let $W = (0, +\infty) \subset \mathbb{R}$. Then W is not a subspace of $\mathbb{R}$ (it is not closed under scalar multiplication). On the other hand, it is not hard to see that if we define for $v, w \in W$ and $\alpha \in \mathbb{R}$ addition and scalar multiplication on W by

$$\alpha \cdot v = v^\alpha$$
$$v + w = vw,$$

then W is a vector space (with these operations). The zero vector in W is the element 1.
(2) is true. Indeed, $\text{span}(S)$ is contained in any subspace of V that contains S, since that subspace is closed under addition and scalar multiplication. Thus the intersection is no smaller than $\text{span}(S)$. On the other hand, $S \subset \text{span}(S)$, so the intersection of all subspaces containing S is no bigger than $\text{span}(S)$.
(3) is false. For example, the set $((1,0),(2,0),(0,1))$ of vectors in $\mathbb{R}^2$ is linearly dependent; however, the last vector is not a linear combination of the first two.

Problem 3. Let $\{v_1, \ldots, v_n\}$ be a linearly independent set of vectors in V. Let $\{u_1, \ldots, u_m\}$ be another linearly independent set of vectors in V. Suppose that $n < m$. Show that the vectors $\{v_1, \ldots, v_n\}$ can not form a basis of V.

Solution. Since $\{u_1, \ldots, u_m\}$ are linearly independent, the set they form can be completed, if necessary, to form a basis for V, $\{u_1, \ldots, u_m, w_1, \ldots, w_k\}$. Thus $\dim V = m + k > n$. Thus any basis for V must have exactly $m + k > n$ elements. Thus $v_1, \ldots, v_n$ cannot be a basis for V.

Problem 4. Let $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ be given by

$$T(f(x)) = (f(0), f'(0))$$

and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$U(a, b) = (a + b, a - b)$$

Let $\alpha = \{1, x, x^2\}$ be a basis of $P_2(\mathbb{R})$ and $\beta = \{(1, 0), (0, 1)\}$ be a basis of $\mathbb{R}^2$. Compute the matrix $[U \circ T]_\alpha^\beta$ of the composition of T and U.

Solution. We have:

$$\begin{aligned} T(1) &= (1, 0) \\ T(x) &= (0, 1) \\ T(x^2) &= (0, 0). \end{aligned}$$

Let $S = U \circ T$. Then

$$\begin{aligned} S(1) &= (1, 1) = 1(1, 0) + 1(0, 1) \\ S(x) &= (1, -1) = 1(1, 0) + (-1)(0, 1) \\ S(x^2) &= (0, 0) = 0(1, 0) + 0(0, 1). \end{aligned}$$

Thus the matrix of S is

$$[S]_\alpha^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Problem 5. Let $f : A \to B$ be a function from a set A to a set B .

1. What does it mean that f is one-to-one (give a definition);
2. Prove that a linear transformation $T : V \to W$ is one-to-one if and only if $N(T) = \{\overrightarrow{0}\}$.

Solution. (1) a function $f : A \to B$ is called one-to-one if $f(x) = f(y)$ implies that $x = y$, for all $x, y \in A$. Equivalently, for each $y \in f(A)$, the set $\{x \in A : f(x) = y\}$ has exactly one element. Equivalently, for all $y \in B$, the set $\{x \in A : f(x) = y\}$ has at most one element. Equivalently, there exists a function $g : f(A) \to A$, so that $g \circ f : A \to A$ equals the identity function. Equivalently, there exists a function $g : B \to A$, so that $g \circ f$ is the identity function. (Any of these equivalent definitions would have sufficed as an answer to the problem).

(2) Assume that $T : V \to W$ is one-to-one. Assume for contradiction that $N(T)$ is nonzero. Then there exists $v \in N(T)$ so that $v \neq \overrightarrow{0}$. But then $T(v) = \overrightarrow{0}$, since $v \in N(T)$ and so

$$T(v) = T(\overrightarrow{0}),$$

which contradicts the assumption that T is one-to-one.

Assume now that $N(T) = \{\overrightarrow{0}\}$. Assume for contradiction that T is not one-to-one. Thus there are two vectors $v, w \in V$, so that $T(v) = T(w)$, but $v \neq w$. But then $v - w \neq \overrightarrow{0}$, and $T(v - w) = T(v) - T(w) = \overrightarrow{0}$. Thus $v - w \in N(T)$ and so $N(T) \neq \{\overrightarrow{0}\}$, contradiction.