PRACTICE MIDTERM (MATH 115A)

Problem 1. Label the following statements as true or false. You will receive 4 points for each correct answer, -4 for each incorrect one, and 0 if you give no answer.

(1) The union of two subspaces of a vector space is a subspace.
(2) A subset of a linearly independent set is linearly independent.
(3) For any $x_1, x_2 \in V$ and $y_1, y_2 \in W$ there is a linear transformation $T : V \to W$ such that $T(x_1) = y_1$, $T(x_2) = y_2$.
(4) For a matrix A the condition $A^3 = O$ implies that $A = O$. (Here O is the zero matrix).
(5) Two vector spaces of different dimensions can not be isomorphic.

Problem 2. Let $V = \mathbb{R}^n$, and let $T : V \to V$ be a map. Consider the subset of $\mathbb{R}^{2n}$ defined by

$$G = \{(v, w) : \, v, w \in \mathbb{R}^n, v = T(w)\}.$$

(This subset is called the *graph* of T). Show that T is linear if and only if G is a subspace of $\mathbb{R}^{2n}$.

Problem 3. Let u, v, w be three distinct vectors in a vector space V. Show that if $\{u, v, w\}$ is a basis, then so is the set $\{u+v+w, v+w, w\}$.

Problem 4. Is there a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, so that

$$\begin{aligned} T(1,2) &= (3,4,5) \\ T(6,7) &= (8,9,10)? \end{aligned}$$

If yes, compute $T(11, 12)$. If not, explain why not.

Problem 5. Let $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ be given by

$$T(A) = \mathrm{tr}(A).$$

(a) Find a basis for the null space of T.
(b) Find the dimensions of the null space and the range of T.
(c) Let α be the standard basis for $M_{2\times 2}(\mathbb{R})$ and let γ be the standard basis for $\mathbb{R}$. Find the matrix $[T]_\alpha^\gamma$.