Math 115a
Hunter Summer 2012

Midterm

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Signature:

Instructions:

- There are 6 problems. Make sure you are not missing any pages.
- You may use without proof anything proven in the sections of the book covered by this test.
- You may only cite an exercise from the book if it was assigned as homework.
- No calculators, phones, books, or notes are allowed.

Question	Points	Score
1	10	
2	15	
3	15	
4	20	
5	15	
6	10	
Total:	~~80~~	

85

1. (10 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(a) Find a basis for the null space $null(T)$. (7 points)
(b) What is the rank and nullity of T? (3 points)

null(T) = set of vector mapped to zero

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \\ 3x_1 + x_2 + 4x_3 = 0 \end{cases}$$ (2)

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ 0 - x_2 - x_3 = 0 \quad \text{subtracting } 2 * \text{ eqn } 1 \\ 0 - 2x_2 - 2x_3 = 0 \quad \text{subtracting } 3 \text{ eqn } 1 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 = 0 \Rightarrow x_1 = -x_3 \\ -x_2 = x_3 \\ 0 = 0 \end{cases}$$ (1)

so vectors of the form $\begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix}$ (2) are in the null space of T

Hence, $\boxed{\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\} \text{ (2) is a basis for the null space of } T.}$

the nullity of T is the dim(null(T)) = the (number of basis (vectors of null(T)) = 1 $\Rightarrow$

$\boxed{\text{nullity}(T) = 1}$ (1)

by the Dimensionality theorem

$\dim(V) = \text{nullity}(T) + \text{rank}(T)$ (1)

so $3 = \dim(\mathbb{R}^3) = 1 + \text{rank}(T)$

$\Rightarrow \boxed{\text{rank}(T) = 2.}$ (1)

2. **(15 points) Prove (using induction) that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W then $a_1w_1 + a_2w_2 + \ldots + a_nw_n \in W$ for any scalar $a_1, a_2, \ldots, a_n$.**

Proof. Let $W \subset V$ a ~~vector space~~ subspace and let

$w_1, w_2, \ldots, w_n \in W$ then

w.t.s $a_1w_1 + a_2w_2 + \ldots + a_nw_n \in W$

for all $a_1, a_2, \ldots, a_n \in \mathbb{F}$.

Proof by induction, show $P(1)$ is true, (2)

$P(1)$ or $P(2)$ = $a_1, a_2 \in \mathbb{F}$ and $w_1, w_2 \in W$ then

w.t.s. $a_1w_1 + a_2w_2 \in W$.

$a_1w_1 \in W$ and $a_2w_2 \in W$ because W is a subspace it is closed under scalar multi. (1)

and $a_1w_1 + a_2w_2 \in W$ because W is a (1) subspace and $(x, y \in W \Rightarrow x+y \in W)$.

thus $P(2)$ is true.

Now assume $P(k)$ is true, i.e. (3)

if $w_1, \ldots, w_k \in W$ and $a_1, \ldots, a_n$ scalars

$a_1w_1 + \ldots + a_kw_k \in W$.

w.t.s $P(k+1)$ is true, i.e. $a_1w_1 + \ldots + a_{k+1}w_{k+1} \in W$ (2)

Let $w_1, \ldots, w_k, w_{k+1} \in W$, $a_1, \ldots, a_k, a_{k+1}$ scalars

then $a_1w_1 + \ldots + a_kw_k \in W$ by assumption.

so $(a_1w_1 + \ldots + a_kw_k) + a_{k+1}w_{k+1}$ is $\in W$ (2)

if $a_{k+1}w_{k+1}$ is in W.

(2) $a_{k+1}w_{k+1}$ is in W, because W is a subspace.

thus $(a_1w_1 + \ldots + a_kw_k) + a_{k+1}w_{k+1} \in W$ since W is a subspace

and $x, y \in W \Rightarrow x+y \in W$

Hence $a_1w_1 + \ldots a_{k+1}w_{k+1} \in W$. i.e. $P(k+1)$ is true.

Thus by induction we are DONE! $a_1w_1 + \ldots + a_nw_n \in W$. (1)

3. ~~(10 points)~~ 15 Let V be the vector space $M_{2\times2}(\mathbb{R})$. Let W be the subspace of diagonal matrices.

Let $T\colon V \to V$ be the linear transformation given by

$$T(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} b & c \\ d & a \end{pmatrix}.$$

$W = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ s.t. } a,b \in \mathbb{R} \right\}$

Let

$$U = \{v \in V : T(v) \in W\}.$$

(a) Prove U is a subspace of V. 10 points

(b) Find a basis for U. (You do not need to prove it is a basis.) (5 points)

(b) U is the set of vectors that get mapped to a diagonal matrix i.e. $\left\{ \begin{pmatrix} *_1 & 0 \\ 0 & *_2 \end{pmatrix} : *_i \in \mathbb{R} \right\}$

So, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & c \\ d & a \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$ (2) $\Rightarrow$ $c = 0$, $d = a$

So $U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ s.t. } a,b \in \mathbb{R} \right\}$ (1) with basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ (2)

(a) (2) U is a subset of V (i.e. $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in U \subset M_{2\times2}(\mathbb{R})$)

to show U is a subspace w.t.s.

(2) $0 \in U$, closed under addition, scalar multi.

$T\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ a diagonal matrix $\Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in U$. (2)

Let $u = \begin{pmatrix} u_1 & u_2 \\ 0 & 0 \end{pmatrix} \in U$, $v = \begin{pmatrix} v_1 & v_2 \\ 0 & 0 \end{pmatrix} \in V$ then

$u + v = \begin{pmatrix} u_1+v_1 & u_2+v_2 \\ 0+0 & 0+0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ where $a = u_1+v_1$, $b = u_2+v_2$ $u+v \in U$. (2) i.e. closed under addition

for $a \in \mathbb{R}$ $au = a\begin{pmatrix} u_1 & u_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} au_1 & au_2 \\ 0 & 0 \end{pmatrix} \in U$ i.e. closed under scalar multi.

thus U is a subspace. (2)

$a\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} \notin W$

4. (20 points) Let $U = \{A \in M_{2x2}(\mathbb{R}):\ -A = A^T\}$ be a subset of the vector space $M_{2\times 2}(\mathbb{R})$ of 2×2 skew-symmetric (or antisymmetric) matrix whose transpose is also its negative; i.e., $-a_{i,j} = a_{j,i}$. For example,

$$\begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix},$$

is skew-symmetric.
(a) Find a basis for U. (6 points)
(b) What is the dimension of U? (2 points)
(a) Prove it is a basis U. (12 points)

(a) - show what do vectors of U look like and why?

$U = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}$

a basis for U = $\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ (1)

b) $\boxed{\dim(U) = \#\text{ of basis vectors},\ 1 = 1}$ (+1)

if A is skew-symmetric then

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix} \Rightarrow$ $a = -a$, $b = -c$, $c = -b$, $d = -d$; $a + a = 0 \Rightarrow a = 0$, $b - c = 0$, $c - b = 0$, $d + d = 0 \Rightarrow d = 0$; $b - c = 0$ (2)

a) So skew-symmetric matrices must have this form, $U = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right\}$ $b \in \mathbb{R}$ and $\boxed{\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \text{ is a basis for } U.}$ (2)

Proof: Let $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ be a vector in U. (4)

$\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = -b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ thus $\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ spans U (2)

$\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ is linearly independent because (1)

the only solution to $a\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (2) is $a = 0$, the trival solution.

Hence $\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ is a basis of U. (2)

5. (15 points) Prove the following lemma, Let $T: V \to W$ be a linear transformation, then T is one-to-one if and only if $null(T) = \{O\}$.

($\Leftrightarrow$) = $\Rightarrow$ and $\Leftarrow$

Proof: ($\Rightarrow$) First suppose the linear map $T: V \to W$ is one-to-one, w.t.s. $null(T) = \{0\}$. It's clear $0 \in null(T)$ because $T0 = 0$ (T is linear). (2) Now w.t.s. there is no other element in the $null(T)$. (2)

Suppose not, suppose for contradiction $\exists\, v \in V$, $v \neq 0$ (2) s.t. $v \in null(T)$ i.e. $Tv = 0$, then $Tv = 0 = T0$ but T is one-to-one so $Tv = T0$ forces $v = 0$ a contradiction. (2)

($\Leftarrow$) Now suppose $null(T) = \{0\}$ w.t.s. T is one to one i.e. when $Tv = Tv'$ then $v = v'$. (2)

So let $Tv = Tv'$ then $Tv - Tv' = 0$ so $T(v - v') = 0$ because T is linear. (3) Thus $v - v' \in null(T)$, hence by the hypothesis ($null(T) = \{0\}$) $v - v' = 0$, so $v = v'$ as desired. (2)

Therefore $T: V \to W$ (the linear tranformation) T is one-to-one iff $null(T) = \{0\}$.

$(a_1-a_2, a_1, 2a_1+a_2)$

6. (10 points) (a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, [illegible])$. Let β be the standard ordered basis $\{e_1, e_2\}$ for $\mathbb{R}^2$ and $\alpha = \{(1,1,0), (0,1,1), (2,2,3)\}$, compute the matrix representation $[T]_\beta^\alpha$. (5 points)

$T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$

$T(1,0) = (1, 1, 2) = -\frac{1}{3}(1,1,0) + 0 + \frac{2}{3}(2,2,3)$ (1.5)

$T(0,1) = (-1, 0, +1) = x(1,1,0) + y(0,1,1) + z(2,2,3)$ (2)

so $\begin{cases} -1 = x + 2z \\ 0 = x + y + 2z \\ 1 = y + 3z \end{cases} \Rightarrow \begin{cases} x = -2z - 1 \\ 0 = (-2z-1) + y + 2z \\ 1 = y + 3z \end{cases}$ $\boxed{x = -\frac{1}{3},\ y = -1,\ z = \frac{2}{3}}$

$\Rightarrow T(0,1) = (-1, 0, 1) = -\frac{1}{3}(1,1,0) - 1(0,1,1) + \frac{2}{3}(2,2,3)$ (1)

$$[T]_\beta^\alpha = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} \\ 0 & -1 \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$ (1)

(b) A transformation $T\colon V \to W$ is called *linear* if what? (3 points)

$\forall v_1, v_2 \in V$ ~~then~~ $T(v_1 + v_2) = T(v_1) + T(v_2)$

$\forall v_1 \in V\ \forall a \in \mathbb{F}\ \ T(a v_1)$ (3)

(c) Let $T\colon V \to W$ be a linear tranformation define the *inverse* of T. (2 points)

The linear transformation $S: W \to V$ is the inverse of T if $TS = I_W$ and $ST = I_V$, (2)

(and call the inverse T^{-1})