Math 115a
Hunter Summer 2012

Final Exam

Name: Solutions

Signature:

Instructions:

- There are 6 problems. Make sure you are not missing any pages.
- You may use without proof anything proven in the sections of the book covered by this test.
- You may only cite an exercise from the book if it was assigned as homework.
- No calculators, phones, books, or notes are allowed.

Question	Points	Score
1	15	
2	15	
3	20	
4	15	
5	20	
6	15	
Total:	100	

11:05 31 student

0a Every finite dim ips space every linear functional can be written as an inner product.

1. **(15 points)** (a) **State the Riesz Representation Theorem (6 points)**
(b) **Prove the Riesz Representation Theorem (9 points)**

Riesz Rep. Thm. Let V be a finite dim inner product space (1) (2) and let $T : V \to \mathbb{R}$ be a linear functional on V. Then $\exists\, w \in$ [illegible] st. $Tv = \langle v, w \rangle$ (3) $\forall\, v \in V$. (1)

Proof Let V be finite dim. inner product space. By the Gram-Schmidt process we can find an orthonormal basis $v_1, \dots, v_n$ for V. (1) For $v \in V$ $\quad v = \langle v, v_1 \rangle v_1 + \dots + \langle v, v_n \rangle v_n$ (1)

Applying T gives,
$$Tv = T(\langle v, v_1 \rangle v_1 + \dots + \langle v, v_n \rangle v_n) \quad (1)$$
$$= T\langle v, v_1 \rangle v_1 + \dots + T\langle v, v_n \rangle v_n \quad (1)$$
$$= \langle v, v_1 \rangle Tv_1 + \dots + \langle v, v_n \rangle v_n \quad (1)$$

Since $Tv_1, Tv_2, \dots, Tv_n \in F$ (1) and $\langle v, w \rangle c = \langle v, \bar{c} w \rangle$ (1)
$$= \langle v, \overline{Tv_1}\, v_1 \rangle + \dots + \langle v, \overline{Tv_n}\, v_n \rangle$$
$$= \langle v, \overline{Tv_1}\, v_1 + \dots + \overline{Tv_n}\, v_n \rangle \quad (1)$$

so let $w \in V$ $\quad w := \overline{Tv_1}\, v_1 + \dots + \overline{Tv_n}\, v_n$ (1)

then $Tv =$ [illegible] $= \langle v, w \rangle \quad \forall\, v \in V$. ∎

2. (15 points) (a) Define eigenvalues and eigenvectors. (3 points)
(b) Prove if A and B are similar $n \times n$ matrices, then A and B have the same eigenvalues. (12 points)

a) Let $A \in \mathbb{C}^{n\times n}$. λ is an eigenvalue of A if $\exists\, v \neq 0$ st. $Av = \lambda v$ and v is called the eigenvector corresponding to λ.

b) A is similar to B iff $A = QBQ^{-1}$ (3) then

The charac. poly. $f(\lambda) = \det(A - \lambda I) \overset{(3)}{=} \det(QBQ^{-1} - \lambda I)$

note; $QQ^{-1} = I$

so $I \cdot I = QQ^{-1} I$
$= QIQ^{-1}$

note: ~~$1 = \det(I) = \det(QQ^{-1}) = \det(Q^{-1}) = \det(Q)$~~

$$= \det(QBQ^{-1} - \lambda QIQ^{-1}) \quad (2)$$
$$= \det(Q(B - \lambda I)Q^{-1}) \quad (2)$$
$$= \det(Q) \cdot \det(B - \lambda I) \det(Q^{-1})$$
$$(2) = \det(Q) \cdot \frac{1}{\det(Q)} \cdot \det(B - \lambda I)$$
$$= 1 \cdot \det(B - \lambda I)$$
$$= \det(B - \lambda I) \quad (1)$$

thus A and B have the same characteristic polynomial ie.

A and B have the same eigenvalues.

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or let $Av = \lambda v$ and $A = QBQ^{-1}$

thus
$QBQ^{-1}v = \lambda v$
$Q^{-1}QBQ^{-1}v = Q^{-1}\lambda v$
$BQ^{-1}v = \lambda Q^{-1}v$
let $Q^{-1}v = w$
thus $Bw = \lambda w$ ie.
λ is an eigenvalue of B
and λ is an eigenvalue of A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

3. (20 points) (a) Find the inverse of $A = $ [illegible]. (5 points)

(b) A is invertible iff $det(A) = 0$ iff the only solution to $Ax = 0$ is the trivial solution, state three additional equivalences to A is invertible. (5 points)

(c) Let V be an inner product over $\mathbb{R}$. Let W be a subspace of V. Define $W^{\perp}$ to be the subspace of all $x \in V$ such that $< x, y >= 0$ for all $y \in W$ i.e.

[illegible]

Prove $W \cap W^{\perp} = \{0\}$. (9 points)

$W^{\perp} = \{x \in V \text{ s.t. } \langle x, y \rangle = 0 \ \forall y \in W\}$

a)

$$[A | I] = \left[\begin{matrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & -1 & -1 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \to \left[\begin{matrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -3 & -4 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right]$$

(1)

$$\to \left[\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} -5 & 3 & 0 \\ 1/3 & 0 & -1/3 \\ 2 & -1 & 0 \end{matrix}\right] \to \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} -1/3 & 1/3 & 2/3 \\ -7/3 & +4/3 & -1/3 \\ 2 & -1 & 0 \end{matrix}\right]$$

$-\frac{4}{3}$ $\quad -\frac{8}{3} + \frac{4}{3}$ $\quad +\frac{14}{3}$

$$\to [I | A^{-1}] \quad (2) \qquad \boxed{A^{-1} = \begin{bmatrix} -1/3 & 1/3 & 2/3 \\ -7/3 & 4/3 & -1/3 \\ 2 & -1 & 0 \end{bmatrix}} \quad (2)$$

b) (2) points each

1. $Ax = b$ has exactly one solution $\forall b \in \mathbb{F}^n$
2. the rows (columns) of A are linearly indep.
3. zero is not an eigenvalue.
4. $T: \mathbb{F}^n \to \mathbb{F}^n$, $T(x) = Ax$ $\forall x \in \mathbb{F}^n$ is 1-1 and onto

c)

Proof. wts. $0 \in W \cap W^{\perp}$ i.e. $0 \in W$ and $0 \in W^{\perp}$.
W is a subspace $\Rightarrow 0 \in W$ (2). $0 \in W \Rightarrow \langle 0, y \rangle = 0 \ \forall y \in W$
$\Rightarrow 0 \in W^{\perp}$ (2). Now wts. $\{0\}$ is the only $v \in W \cap W^{\perp}$ (1).
Assumenot, assume $\exists v \neq 0 \in W \cap W^{\perp}$ i.e. $v \neq 0 \in W$ and $v \in W^{\perp}$ (2)
so $v \in W^{\perp} \Rightarrow \langle v, y \rangle = 0 \ \forall y \in W$ (2) and $v \in W \Rightarrow$ (2)
$\langle v, v \rangle = 0 \Rightarrow v = 0$ (by def of inner product) this is a contradiction $\Rightarrow \{0\} = W \cap W^{\perp}$.

4. (15 points) Let $T\colon V \to W$ be the linear transformation from one inner product space to another. And let $T^*\colon W \to V$ where

$$\langle Tv, w\rangle = \langle v, T^*w\rangle .$$

(a) T^* is call the what of T? (3 points) $\boxed{\text{adjoint}}$

(b) Prove T^* is linear. (12 points)

b) w.t.s. T^* is linear i.e. $T^*(x+y) = T^*(x) + T^*(y)$

and $T^*(cx) = cT^*(x)$ or $T^*(cx+y) = cT^*(x) + T^*(y)$ $\forall x,y \in V$ and $\forall c \in \mathbb{F}$

Let $v, x, y \in V$ and $c \in \mathbb{F}$

$$\begin{aligned}\langle v, T^*(cx+y)\rangle &= \langle Tv, cx+y\rangle \\ &= \langle Tv, cx\rangle + \langle Tv, y\rangle \\ &= \bar{c}\langle Tv, x\rangle + \langle Tv, y\rangle \\ &= \bar{c}\langle v, T^*x\rangle + \langle v, T^*y\rangle \\ &= \langle v, cT^*x\rangle + \langle v, T^*y\rangle \\ &= \langle v, cT^*(x) + T^*(y)\rangle, \ \forall v \in V\end{aligned}$$

$\Rightarrow T^*(cx+y) = cT^*(x) + T^*(y)$ $\forall x,y \in V$ and $c \in \mathbb{F}$ ■

5. (20 points) Let $A \in \mathbb{C}^{3\times 3}$, $A = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 2 \end{pmatrix}$. And let $B = AA^* = \begin{pmatrix} 2 & 2i \\ -2i & 5 \end{pmatrix}$, with eigenvectors $\begin{pmatrix} i \\ 2 \end{pmatrix}$ corresponding to $\lambda = 6$ and $\begin{pmatrix} -2i \\ 1 \end{pmatrix}$ corresponding to $\lambda = 1$.

(a) Find the diagonalization of B. (7 points)

(b) Find the singular value decompostion of A (you must compute additional eigenvalues, eigenvectors). (13 points)

a.) $\boxed{B = QDQ^{-1}} \rightarrow \boxed{Q = \begin{bmatrix} i & -2i \\ 2 & 1 \end{bmatrix}}$ $\boxed{D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}}$

$[Q \mid I] = \left[\begin{array}{cc|cc} i & -2i & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -2 & -i & 0 \\ 0 & -3 & +3i & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -3i & -2/3 \\ 0 & 1 & -i & -1/3 \end{array}\right]$

$\to [I \mid Q^{-1}] \Rightarrow \boxed{Q^{-1} = \begin{bmatrix} -3i & -2/3 \\ -i & -1/3 \end{bmatrix}}$

$D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$

b.) $A^*A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -i & 2 \end{pmatrix} \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 2 \\ -i & 2 & 5 \end{bmatrix}$

$\lambda_1 = 6,\ \lambda_2 = 1,\ \lambda_3 = 0$

$\lambda_1 = 6 \quad (A - \lambda I)x = 0 \Rightarrow \begin{bmatrix} -5 & 0 & i \\ 0 & -5 & 2 \\ -i & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{bmatrix} i \\ 2 \\ 5 \end{bmatrix},\ \|v_1\| = 30$

$\lambda_2 = 1 \Rightarrow \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 2 \\ -i & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{pmatrix} -2i \\ 1 \\ 0 \end{pmatrix}\ \|v_2\| = 5$

$\lambda_3 = 0 \Rightarrow \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 2 \\ -i & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_3 = \begin{bmatrix} -i \\ -2 \\ 1 \end{bmatrix} \quad \|v_3\| = 6$

6. **(15 points) Prove the following lemma, Let $T\colon V \to W$ be a linear transformation, then T is one-to-one if and only if $null(T) = \{O\}$.**

Proof. ($\Rightarrow$) First suppose the linear map $T: V \to W$ is one to one, wts. $null(T) = \{0\}$. It's clear $0 \in null(T)$ because $T(0) = 0$, T linear. So wts. there is no other element in the null space of T. Suppose not, suppose for contradiction $\exists v \in V$ $v \neq 0$ s.t. $v \in null(T)$. i.e. $Tv = 0$ then $Tv = 0 = T0$ but T is one-to-one so $Tv = T0$ forces $v = 0$ a contradiction.

($\Leftarrow$) Now suppose $null(T) = \{0\}$ wts. T is one to one. i.e. when $Tv = Tv'$ then $v = v'$. So let $Tv = Tv'$ then $Tv - Tv' = 0$, since T is linear $T(v - v') = 0$, thus $v - v' \in null(T) = \{0\}$ so $v - v' = 0$ thus $v = v'$, as desired.

Hence the linear transformation is one to one iff $null(T) = \{0\}$. ∎