Math 115A HW4 Solutions

September 5, 2012 University of California, Los Angeles

1 Problem 4.1.3b

Calculate the determinant,

$$\begin{bmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{bmatrix}$$

Solution: The textbook's instructions give us,

$$(5-2i)7i - (6+4i)(-3+i) = 35i + 14 - (-22-6i) = 36 + 41i$$

2 Problem 4.1.7

Prove that $\det(A^t) = \det(A)$ for any $A \in M_{2\times 2}(F)$.

Proof: Write out,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

so that,

$$A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Then we have $\det(A) = ad - bc = ad - cb = \det(A^t)$.

3 Problem 4.1.9

Prove that $\det(AB) = \det(A) \cdot \det(B)$ for any $A, B \in M_{2\times 2}(F)$.

Proof: Since we are just dealing with 2×2 matrices, we can just write out what AB is and take its determinant, showing that it will be the same as $\det(A) \cdot \det(B)$. We have,

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

The determinant of this is:

$$(a_{11}b_{11} + a_{12}b_{21}) \cdot (a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22}) \cdot (a_{21}b_{11} + a_{22}b_{21}) \tag{1}$$

But we also have,

$$\det(A) \cdot \det(B) = (a_{11}a_{22} - a_{12}a_{21}) \cdot (b_{11}b_{22} - b_{12}b_{21})$$

Multiplying this out clearly gives (1).

4 Problem 4.2.9

Evaluate the determinant along the third row:

$$\begin{bmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{bmatrix}$$

Solution: We proceed carefully, taking the correct signs into account, yielding,

$$3\det\begin{bmatrix} 1+i & 2 \\ 0 & 1-i \end{bmatrix} - 4i\det\begin{bmatrix} 0 & 2 \\ -2i & 1-i \end{bmatrix} + 0\det\begin{bmatrix} 0 & 1+i \\ -2i & 0 \end{bmatrix} = 6 + 16 = 22$$

5 Problem 4.2.14

Calculate the determinant,

$$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 0 \\ 7 & 0 & 0 \end{bmatrix}$$

Solution: There are a bunch of zeros in the third row and third column. The last problem asked us to calculate the determinant along the third row, so let's do the third column. Again, we check to see when to multiply by -1 and we have (since there are 0's in the second and third entries of the last column),

$$4\det\begin{bmatrix} 5 & 6 \\ 7 & 0 \end{bmatrix} = -168$$

6 Problem 4.3.9

Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.

Proof: Let Ω denote the set of all invertible matrices. Suppose $A \in M_{n\times n}(F)$ and is triangular. We must first prove the following,

Lemma: If A is triangular, then,

$$\det(A) = \prod_{i=1}^{n} A_{ii} \tag{2}$$

(That is, the determinant is the product of the diagonal entries.)

Proof: If A is 1×1, the determinant is simply the A_{11} entry. Suppose that (2) holds for all $n-1 \times n-1$ matrices. Suppose then that A is $n \times n$ and suppose also, without loss of generality, that A is upper triangular (if not, then by theorem 4.8, $\det(A^t) = \det(A)$). We have a matrix that looks something like this,

$$\begin{bmatrix} A_{11} & & * \\ & \ddots & \\ 0 & & A_{nn} \end{bmatrix}$$

Notice that I used a $*$ to denote the nonzero entries above the diagonal. This is because these entries will become largely irrelevant momentarily. Well, let's expand along the first column (since the determinant does not depend on how we calculate it). Then what results is,

$$\det(A) = A_{11} \det A^*$$

where A^* denotes the matrix consisting of all entries A_{ij} where $2 \leq i, j \leq n$. But the A^* matrix is $n-1 \times n-1$, so its determinant follows (2), except the limit is $n-1$ instead of n. Thus, what results is,

$$\det(A) = A_{11} \prod_{i=1}^{n-1} A_{ii}$$

which is exactly (2). The result follows by induction.

Now, by the lemma, if $A \in \Omega$, it suffices to show that $\det(A) \neq 0$. But this follows by a corollary to theorem 4.7 in the textbook. Thus, no A_{ii} entry can possibly be zero. On the other hand, if all $A_{ii} \neq 0$, then by the lemma, $\det(A) \neq 0$. By the same corollary, $A \in \Omega$.

7 Problem 4.3.11

Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

Proof: Well, M is skew-symmetric, so, $M^t = -M$. But we also know that $\det(M) = \det(M^t)$, so $\det(M) = \det(-M)$. But also, $\det(-M) = (-1)^n \det M$. Since n is odd, we conclude that $\det M = -\det M$, which is not possible unless $\det M = 0$. Ergo, $M \notin \Omega$ (notation from previous problem).

Notice, however, that if n is even, M may be invertible. Consider, for instance,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

It's determinant is 1, so it is invertible.

8 Problem 4.3.13b

Prove that if Q is a unitary matrix, then $|\det(Q)| = 1$.

Proof: By the previous result (which you were not required to show), $\det(\overline{M}) = \overline{\det(M)}$. But we also know that,

$$\det(QQ^*) = \det(I)$$

Therefore, we have,

$$\det(Q)\det(Q^*) = 1$$

But this is,

$$\det(Q)\overline{\det(Q^t)} = \det(Q)\overline{\det(Q)} = |\det(Q)| = 1$$

Done. If the last step isn't clear, if $c = a + bi$, then $c\bar{c} = (a + bi)(a - bi) = a^2 + b^2 = |c|$.

9 Problem 5.1.3c

See text for instructions. The matrix given is,

$$\begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}$$

Solution: Let A be the matrix. Then $\det(A - \lambda I) = \lambda^2 - 1$. Hence, the eigenvalues are 1 and -1. Now, we find a vector b such that,

$$\begin{bmatrix} i - 1 & 1 \\ 2 & -i - 1 \end{bmatrix} b = 0$$

It is apparent that a vector that will work is,

$$\begin{bmatrix} 1 \\ -i + 1 \end{bmatrix}$$

We do the same thing for the eigenvalue 1 and get the following eigenvector,

$$\begin{bmatrix} 1 \\ -i-1 \end{bmatrix}$$

What emerges then is that a basis for C^2 is this,

$$\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ -i+1 \end{bmatrix}, \begin{bmatrix} 1 \\ -i-1 \end{bmatrix} \right\}$$

since these vectors are clearly linearly independent. The matrix Q that we want then is,

$$\begin{bmatrix} 1 & 1 \\ -i-1 & -1+1 \end{bmatrix}$$

And we are done.

10 Problem 5.1.14

For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

Proof: We have often used the property that $\det A = \det A^t$ and we will use this again. The characteristic polynomial, $P(\lambda)$ for a matrix A is given by,

$$P(\lambda) = \det(A - \lambda I) =$$

But notice,

$$\det(A - \lambda I) = \det((A - \lambda I)^t)$$

Since I is diagonal, $(A - \lambda I)^t = A^t - \lambda I$, so we get the desired result,

$$\det(A - \lambda I) = \det(A^t - \lambda I)$$