1 Problem 2.2.10

Let V be a vector space with the ordered basis $\beta = \{v_1, ..., v_n\}$. Define $v_0 = 0$. By Theorem 2.6, there exists a linear transformation $T : V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for all $j = 1, 2, ..., n$. Compute $[T]_\beta$.

Solution: Well, we want to set up something like this,

$$\begin{bmatrix} [T(v_1)]_\beta & [T(v_2)]_\beta & ... & [T(v_n)]_\beta \end{bmatrix}$$

Computing these is straightforward, given that $v_0 = 0$ (for this yields $T(v_1) = v_1$). The $T(v_j)$ entry has a one in the jth column component and in the $j - 1th$ column component. This yields,

$$\begin{bmatrix} 1 & 1 & 0 & ... & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

2 Problem 2.3.10

Let A be $n \times n$. Prove that A is diagonal if and only if $A_{ij} = \delta_{ij} A_{ij}$ for all i and j.

Proof: If A is diagonal, then all entries other than those in the diagonal are necessarily 0. Let A_{ij} be an entry of A. Then since,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

then if A is diagonal, it certainly holds that $A_i j = \delta_i j A_i j$. Now, if $A_i j = \delta_i j A_i j$, then the only entries that are not necessarily zero are in those where $i = j$, namely, the diagonal entries. Hence, A is diagonal.

3 Problem 2.3.12

Let V, W, Z be vector spaces and let $T : V \to W$ and $U : W \to Z$ be linear.

1. Prove that if UT is one to one, then T is one to one. Must U also be one to one?

2. Prove that if UT is onto, then U is onto. Must T also be onto?

3. Prove that if U and T are one to one and onto, then UT is also.

Proof: For 1), let $v \in \ker T$. Then $UT(v) = U(T(v)) = 0$. Hence, $v \in \ker(UT)$, so UT is one to one. This shows that $\ker T \subset \ker(UT) = \{0\}$. This is enough to show that T is one to one. Notice, however, that U may not be one to one. For, consider $T : F^1 \to F^2$ where $T(a) = (a, 0)$ and $U : F^2 \to F^1$ where $U(a, b) = (a)$. Then $UT(a) = a$ for all a so that UT is one to one but $\ker U = \{(0, b) : b \in F\}$. Hence, U is not one to one.

Now, for 2), let $z \in Z$. We must show that $z \in R(U)$. Since UT is onto, there is a $v \in V$ so that $UT(v) = z$. But then $z = UT(v) = U(T(v))$ by definition, so $z \in R(U)$. Notice, however, that T doesn't need to be onto. For, consider the counterexample in part 1). UT is onto, but T isn't since $R(T) = \{(a, 0) : a \in F\} \neq F^2$.

Finally, we prove 3). Let $V \in \ker UT$. Then $T(v) \in \ker U$ by definition: $0 = UT(v) = U(T(v))$. But U is one to one which means that U has a kernel of zero only. So $T(v) = 0$ so $v \in \ker T = \{0\}$ as T is one to one. Thus, $v = 0$ and $\ker UT = \{0\}$.

4 Problem 2.3.13

Let A and B be $n \times n$ matrices. Prove that $\mathrm{tr}AB = \mathrm{tr}BA$ and $\mathrm{tr}A = \mathrm{tr}A^t$.

Proof: The second fact is quite clear; for, $A_{ii} = A^t_{ii}$. Since by definition,

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

we see that the second fact holds. For the first, we need to utilize the definition of matrix multiplication. By definition, the product of A and B is given by these entries,

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

for $1 \leq i, j \leq n$. But notice,

$$(AB)_{ii} = \sum_{k=1}^{n} A_{ik}B_{ki}$$

Also,

$$(BA)_{ii} = \sum_{k=1}^{n} B_{ik}A_{ki}$$

Then we have that,

$$\text{tr}AB = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki}$$

$$\text{tr}BA = \sum_{i=1}^{n}\sum_{k=1}^{n} B_{ik}A_{ki}$$

The scalars will now commute, and now what is seen is that all possible combinations of $A_{ik}B_{ki}$ will be in both sums. Hence, the traces are equal.

5 Problem 2.3.16

Let V be a finite-dimensional vector space. Let $T : V \to V$ be linear.

1. If $\text{rank}(T) = \text{rank}(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$.
2. Prove that $V = R(T^k) + N(T^k)$ for some positive integer k.

Proof:

For 1), we need this first:

Lemma: $\ker T = \ker T^2$.

Proof: Choose $v \in \ker T$. Then $Tv = 0$. But also $T^2v = 0$, so $v \in \ker T^2$. This gives that $\ker T \subseteq \ker T^2$. To prove equality, it suffices to show that the dimensions are equal. But this follows from rank-nullity; for by rank-nullity, we have that $\text{nullity}T^2 = V - \text{rk}T^2 = V - \text{rk}T = \text{nullity}T$. Thus, equality is proven, concluding the lemma.

Now, choose $v \in R(T) \cap N(T)$. Then for some $\alpha \in V$, $T\alpha = v$. But also $T^2\alpha = Tv = 0$ since $v \in N(T)$. Hence, $\alpha \in \ker T^2$. But the nullspaces of T and T^2 are equal, so $T\alpha = v = 0$. Thus, their intersection is only in the zero vector. To show that V is the direct sum of the range and nullspace, first notice that $V \supseteq R(T) + N(T)$ since $T : V \to V$. The dimension of their sum is equal to V since,

$$\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T)) = V$$

so equality follows. Finally, we concluded that the intersection of the range and nullspace is zero, so V is the direct sum of the range and nullspace by definition.

Now, we prove 2). First, we claim that $R(T^{k+1}) \subseteq R(T^k)$ for any k. Indeed, choose $v \in R(T^{k+1})$. Then for some α, $T^{k+1}\alpha = v$. But $T^k(T\alpha) = v$, so $v \in R(T^k)$.

Now, assume that no $k \in \mathbb{Z}^+$ exists to give the fact that $R(T^{k+1}) \supseteq R(T^k)$. Then we have strict containment and $\text{rk}T^{k+1} < \text{rk}R(T^k)$ for every k. This gives this infinite chain,

$$... < \mathrm{rk}T^n < ... < \mathrm{rk}T^2 < \mathrm{rk}T$$

But this means that,

$$0 \leq \mathrm{rk}T^k \leq \dim V$$

for all k, so there exist infinitely many distinct k so that the dimension of the range can be squeezed between that of 0 and V. But this contradicts the finiteness of the dimension of V.

We conclude that there exists a $k \in \mathbb{Z}^+$ such that $\mathrm{rk}T = \mathrm{rk}T^2$. Now, the 2) follows the same way as 1) if we replace T by T^k and T^2 by T^{k+1}.

This problem is arguable one of the most beautiful results you have yet shown!

6 Problem 2.4.4

Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: We guess that $(AB)^{-1} = B^{-1}A^{-1}$. Since A and B are invertible, we have,

$$ABB^{-1}A^{-1} = AA^{-1} = I$$

Also,

$$B^{-1}A^{-1}AB = B^{-1}B = I$$

Thus, AB is invertible and it's inverse is $B^{-1}A^{-1}$.

7 Problem 2.4.6

Prove that if A is invertible and $AB = O$, then $B = O$.

Proof: If A is invertible, apply its inverse to the left hand side of both sides:

$$A^{-1}B = A^{-1}O \Rightarrow B = O$$

8 Problem 2.4.7

Let A be $n \times n$.

1. Suppose that $A^2 = O$. Prove that A is not invertible.
2. Suppose that $AB = O$ for some nonzero $n \times n$ B. Could A be invertible? Explain.

Solution: For 1), suppose A is invertible. Let B be its inverse. Then $AB = BA = I$. But notice that,

$$BAA = A = BO = O$$

Then apply B once again and we get that $BA = 0 \neq I$.

This is obviously a problem. Thus, A can't be invertible. Now, suppose $AB = O$ for some B that is $n \times n$. Notice that A cannot be invertible. For if it could and its inverse was C, then $CAB = B = CO = O$. But $B \neq O$. Thus, A isn't invertible.

9 Problem 2.4.15

Let V and W be $n-$dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Proof: First, suppose that T is isomorphic. We want to show that, if $\beta = \{\beta_1, ..., \beta_n\}$,

$$\sum_{i=1}^{n} c_i T(\beta_i) = 0$$

only when all $c_i = 0$. Well, notice that,

$$\sum_{i=1}^{n} c_i T(\beta_i) = T\left(\sum_{i=1}^{n} c_i \beta_i\right) = 0$$

This shows that the linear combination of basis elements is in the kernel of T, which is zero since T is an isomorphism. Thus, since β is a basis for V, all $c_i = 0$. Hence, $T(\beta)$ is linearly independent. Finally, it is, in fact, a basis since their are n elements in $T(\beta)$ and $\dim W = n$.

Now, suppose $T(\beta)$ is a basis for W. Choose $v \in \ker T$. Well, $v = \sum c_i \beta_i$ for some c_i. Therefore, $\sum c_i \beta_i \in \ker T$. But this also means, by linearity, that,

$$\sum_{i=1}^{n} c_i T(\beta_i) = 0$$

Thus, $c_i = 0$ for all i since it was assumed that $T(\beta)$ is a basis. Thus, $v = 0$ and T is injective. Now, choose $v \in W$. Then $v = \sum c_i T(\beta_i)$ for some c_i. But then $v = T(\sum c_i \beta_i)$ and, since $\sum c_i \beta_i \in V$, we have that v is the mapping of some vector in V. Thus, T is surjective and we conclude that T is an isomorphism.

10 Problem 2.4.17

Let V and W be finite dimensional vector spaces and $T : V \to W$ be an isomorphism. Prove that,

1. $T(V_0)$ is a subspace of W.
2. $\dim(V_0) = \dim(T(V_0))$.

Proof: The zero vector is certainly in $T(V_0)$ since $T(0) = 0$. Choose $x, y \in T(V_0)$. Then $x = T(x')$, $y = T(y')$ for $x', y' \in V_0$. But notice that $T(x' + y') = T(x'))T(y') = x + y$. Hence, $x + y \in T(V_0)$ since $x' + y' \in V_0$. The same follows with scalar multiplication: if $c \in F$ and $x \in T(V_0)$, then $cx = cT(x') = T(cx')$ for some $x' \in V_0$. Hence $cx \in T(V_0)$.

Now, we must prove that $\dim(V_0) = \dim(T(V_0))$. Restrict T to the subspace V_0. Call this restriction $T_{V_0} : V_0 \to T(V_0)$. Then by definition, T_{V_0} is surjective and it is still injective, since the nullspace of the map is still zero (it's the same map; we are just applying it to V_0 only). Then T_{V_0} remains an isomorphism and by theorem 2.19, $\dim(V_0) = \dim(T(V_0))$.