Gambler's Ruin and

The Three State Markov Process

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Signature Page

Thesis: Gambler’s Ruin and the Three State Markov Process

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ABSTRACT

Ruin probabilities are determined for a variety of gambler's ruin models. Specifically, ruin probabilities are found in the classical gambler's ruin model augmented to include catastrophe and windfall probabilities. This problem is solved in both a finite time and infinite time setting. In the finite time case, lattice path combinatorics plays a key role to count sample paths of the Markov chain. In the infinite time case, recurrence relations are solved using probability generating functions and the theory of difference equations. The transient probability functions are explicitly determined for the general three state Markov process. These solutions are categorized into three distinct cases and function forms; examples of each type of transient probability function are presented.

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Chapter 1

Gambler's Ruin in Finite Time

Introduction

In this chapter, the gambler's ruin probabilities, in finite time, are determined in two different settings. The gambler's ruin with catastrophes is solved in Section 2 and then the gambler's ruin problem with catastrophes and windfalls is determined in Section 3. We begin with a key lattice path counting result for solving the classic gambler's ruin in finite time. This combinatoric method is extended to solve the gambler's ruin with catastrophes and the gambler's ruin with catastrophes and windfalls, by isolating the classic gambler's ruin within these generalized models. There are two path counting approaches to find the gambler's ruin with catastrophes in finite time. The first method avoids catastrophes and produces a solution that is dependent upon the catastrophe probability only implicitly. The second approach is more direct and is explicitly catastrophe dependent. The second technique is then used to solve the gambler's ruin with catastrophes and windfalls in Section 3.

Section 1-1

Gambler's Ruin in Finite Time

The gambler's ruin problem is over three hundred fifty years old. It can be traced back to conversation letters back and forth between Blaise Pascal and Pierre Fermat, see [1]. Pascal considered the problem so difficult that he doubted whether Fermat would be able to solve it. In Pascal's opinion, see [1], it was more difficult than all the other probability problems they had discussed. The gambler's ruin problem goes as follows: a player starts with a given amount of money, *j*-dollars, and makes a series of one dollar bets. Assume this player either wins a dollar or loses a dollar on each bet. The player's fortune may be visualized as a Markov chain on the state diagram shown below in Figure 1

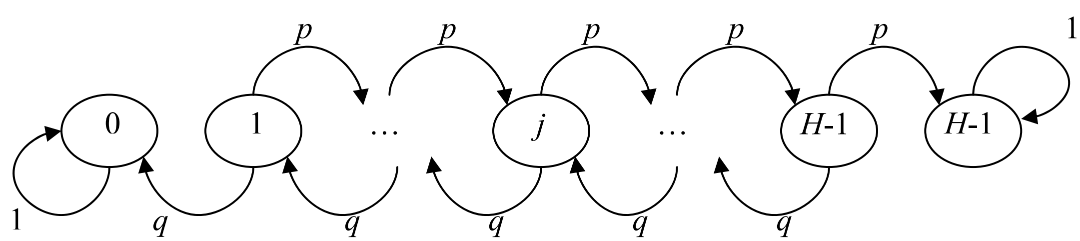

Figure 1 - The state diagram for the gambler's ruin problem

where $0 < p, q < 1$, $p + q = 1$ and $1 \le j \le H - 1$. Each state represents the amount of money the player has at any given time. Starting at state *j*, the player wins or loses a dollar, moving to the left or the right on Figure 1. The game ends when the player loses all of his money or when he reaches his goal of winning *H* dollars. The ruin probability is the chance of reaching state "0" before state "*H*". There are two variants of the gambler's ruin problem: one assumes finite time (or a limited number of bets) and the other assumes an infinite time (or an unlimited number of bets). The solution to both gambler's ruin problems dates back at least to the late 1600's [1,11]. In addition to Pascal and Fermat, solutions were obtained by C. Huygens, J. Bernoulli, A. de Moivre, P. de Montmort and N. Bernoulli, see [1,11]. In this chapter, we analyze the finite time gambler's ruin problem using lattice path combinatorics.

We suppose there is only time for n wagers. The probability of having k dollars after making n bets given you started with j dollars is denoted by $P_{j,k}^{(n)}$ for k=0,1,2,3,…,H. By theory of Markov chains, [4] if

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ q & 0 & p & 0 & & 0 \\ 0 & q & 0 & p & 0 & \\ & & \ddots & & \ddots & \\ 0 & 0 & 0 & 0 & & 1 \end{bmatrix}$$

then P^n has entries $P_{j,k}^{(n)}$. Note: $P_{j,0}^{(n)}$ and $P_{j,H}^{(n)}$ are the probabilities of being absorbed by time n. That is, the probability that by the n^{th} bet the player has no money left is $P_{j,0}^{(n)}$ and $P_{j,H}^{(n)}$ is the probability of meeting his goal of H dollars by the n^{th} bet. At the absorbing states 0 and H there is no place to go but remain there. Let

$$P_{j,k}^{(n)} = \text{Prob(being in state } k \text{ after } n\text{-steps} \mid \text{initially at state } j)$$

as already defined and

$$R_{j,0}^{(n)} = \text{Prob(absorption at state zero in } n\text{-steps} \mid \text{initially at state } j).$$

We next discuss a method to determine an explicit analytic expression for $P_{j,k}^{(n)}$.

For simplicity assume $1 \le j, k \le H-1$. That is, we assume *j, k* are both inside the box in Figure 2.

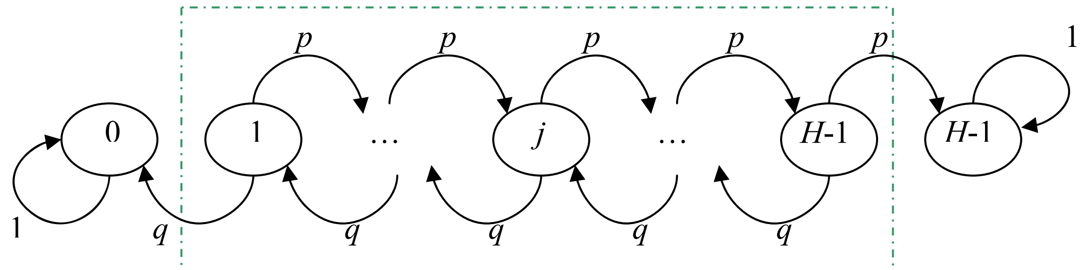

Figure 2

Let $L_{j,k}^{(n)}(H)$ represent the collection all lattice paths going from j to k in n-steps, bounded by horizontal lines y = 0 and y = *H* and restricted to not hit these boundaries.

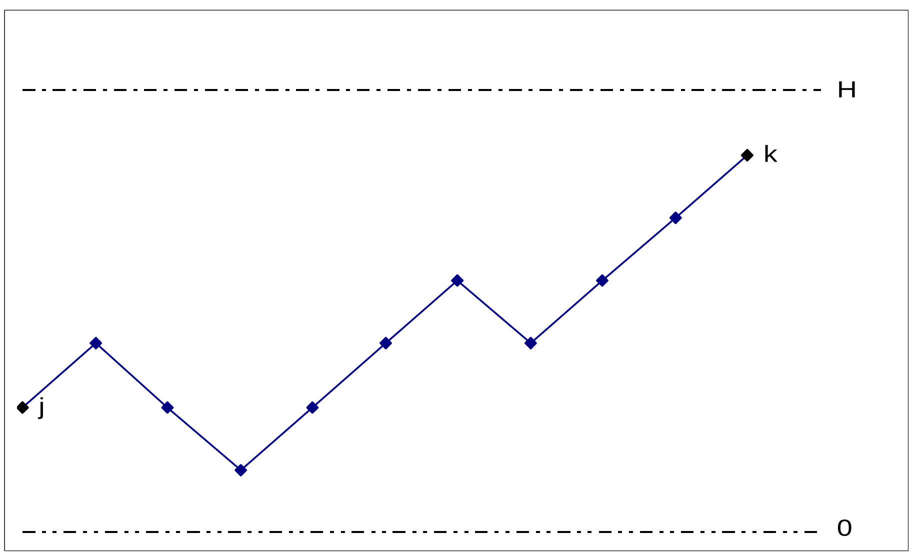

Figure 3 – Typical Lattice Path

To determine $P_{j,k}^{(n)}$, the problem comes down to finding the number of lattice paths going from *j* to *k* in *n*-steps not hitting zero or *H*. The number of lattice paths in $L_{j,k}^{(n)}(H)$ is represented by $\left|L_{j,k}^{(n)}(H)\right|$.

Lemma 1-1.1 For $1 \le j,k \le H-1$

$$\left|L_{j,k}^{(n)}(H)\right| = \sum_{l=1}^{H+1}\left[\binom{n}{\frac{n-k+j}{2} - l(H+1)}_{+} - \binom{n}{\frac{n+k+j-2}{2} + l(H+1)+1}_{+}\right]$$

where the subscript of "+" means $\binom{n}{x}_{+} = \begin{cases} \binom{n}{x} & if \quad 0 \le x \le n \\ 0 & if \quad x > n \;\; or \;\; x < 0 \end{cases}$

This result is derived using combinatorics, the reflection principle, and the method of inclusion/exclusion. The proof may be found in [9, 10].

Rewriting the sum of binomial coefficients in the preceding lemma as nth powers we obtain a result discovered by Cal Poly Pomona Professor Daniel Marcus, see [7]. Afterwards, it was found out that this result goes back many years, cf. [11].

Lemma 1-1.2 For $0 \le a \le m-1$ where a, m are whole numbers

$$\sum_{g \equiv a \bmod m} \binom{n}{g}_{+} = \frac{1}{m}\sum_{u=1}^{m} w^{-ua}(1+w^{u})^{n}$$

and where $w = \exp\left\{\frac{2\pi i}{m}\right\}$, an m^{th} root of unity.

Combining Lemmas 1-1.1 and 1-1.2 and simplifying gives the following important proposition.

Proposition 1-1.3 For $1 \le j, k \le H-1$

$$\left|L_{j,k}^{(n)}(H)\right| = \frac{2}{H}\sum_{u=1}^{H}\sin\left(\frac{u\pi k}{H}\right)\sin\left(\frac{u\pi j}{H}\right)\left[2\cos\left(\frac{u\pi}{H}\right)\right]^{n}$$

Note: it is not obvious the preceding expression is an integer, let alone that it counts the number of lattice paths. We can think of it like the well-known Fibonacci formula. See [6] for the complete details of the proof of this proposition.

From this proposition a corollary follows.

Corollary 1-1.4

Case 1 $1 \le j, k \le H-1$

$$P_{j,k}^{(n)} = \frac{2}{H} p^{\frac{k-j}{2}} q^{\frac{j-k}{2}} \sum_{u=1}^{H}\sin\left(\frac{u\pi k}{H}\right)\sin\left(\frac{u\pi j}{H}\right)\left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{n}$$

Case 2 $1 \le j \le H-1$

$$R_{j,0}^{(n)} = \frac{2}{H} p^{\frac{1-j}{2}} q^{\frac{1+j}{2}} \sum_{u=1}^{H}\left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)-1\right]^{-1}\sin\left(\frac{u\pi j}{H}\right)\sin\left(\frac{u\pi}{H}\right)\left[\left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^{n}-1\right]$$

Proof. For Case 1, notice that

$$P_{j,k}^{(n)} = \left|L_{j,k}^{(n)}(H)\right| p^{u} q^{d}$$

where u is the number of "up" steps and d the number of "down" steps. As before, p is the probability of moving from state i to state i+1, winning a dollar and q is the probability of moving from state i to state i-1, losing a dollar. The number of sample paths, in Figure 3, that start at state j and end at state k in n-steps without hitting the two absorbing states is $\left|L_{j,k}^{(n)}(H)\right|$. Moreover

$$\begin{array}{c} u+d=n \\ u-d=k-j \\ \hline 2u=n+k-j \end{array} \quad \rightarrow \quad \begin{array}{l} u=\dfrac{n+k-j}{2} \\ d=\dfrac{n-k+j}{2} \end{array}$$

where u is the number of upward steps and d is the number of downward steps. The total number of steps made, n, is equal to the number of upward steps and downward steps. The difference in upward steps and downward steps is equivalent to the difference in ending and starting positions.

This gives Case 1, using the formula for $\left|L_{j,k}^{n}(H)\right|$ found in Proposition 1-1.3 .

For Case 2, the probability of starting at state j and going to state 0 in n-steps is equivalent to the probability of starting at state j, going to state one in n-1 steps, then going to zero on the n^{th}-step, or starting at state j, going to state one in n-2 steps, then going to zero on the $(n\text{-}1)^{\text{th}}$-step, or… This is equal to the sum ,over i, of probabilities going from j to 1 in i-steps times q , the probability of moving from 1 to state zero in one step. Hence,

$$R_{j,0}^{(n)}=\sum_{i=1}^{n-1}P_{j,1}^{(i)}\cdot q$$

But $P_{j,1}^{(i)}$ is determined by Case 1 setting k=1. Thus

$$R_{j,0}^{(n)} = \sum_{i=1}^{n-1} \frac{2}{H} p^{\frac{1-j}{2}} q^{\frac{j-1}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{i} \cdot q$$

$$= \frac{2}{H} p^{\frac{1-j}{2}} q^{\frac{j-1}{2}} q \sum_{i=1}^{n-1} \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{i}$$

$$= \frac{2}{H} p^{\frac{1-j}{2}} q^{\frac{j+1}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \sum_{i=1}^{n-1} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{i}$$

Notice that $\sum_{i=1}^{n-1} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{i}$ is a partial geometric series with $r = 2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)$

Thus,

$$\sum_{i=1}^{n-1} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{i} = \left(1 - 2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^{-1} \left[1 - \left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^{n}\right]$$

So,

$$R_{j,0}^{(n)} = \frac{2}{H} p^{\frac{1-j}{2}} q^{\frac{j+1}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \sum_{i=1}^{n-1} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{i}$$

$$R_{j,0}^{(n)} = \frac{2}{H} p^{\frac{1-j}{2}} q^{\frac{j+1}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(1 - 2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^{-1} \left[1 - \left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^{n}\right]$$

This completes the proof of Case 2.

Roulette Example

A roulette player starts with a given amount of money, *j*= 1,2,3 or 4 dollars, and makes a series of one dollar bets. Assume the player either wins a dollar or loses a dollar on each bet of red or black.

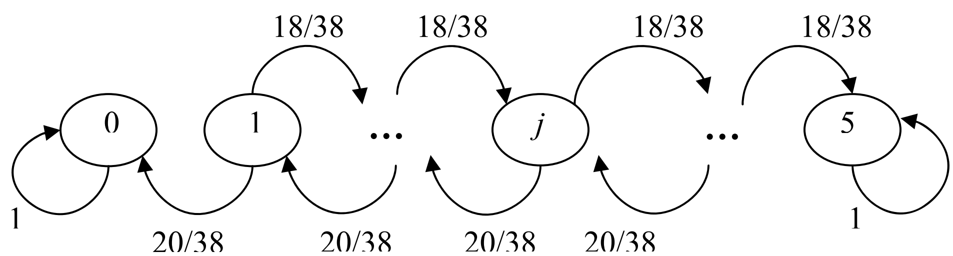

Figure 4

H= 5 (maximum value), *p*=18/38 (probability of a win, placing a bet on black or red), *q*=20/38 (probability of a loss), j= starting amount

$P_{j,k}^{(n)}$ have been calculated in the following table listed as P(j,k).

	j	P(j,0)	P(j,1)	P(j,2)	P(j,3)	P(j,4)	P(j,5)
n = 5							
	1.0000	0.7230	0.0000	0.1472	0.0000	0.0795	0.0503
	2.0000	0.4151	0.1636	0.0000	0.2355	0.0000	0.1858
	3.0000	0.2548	0.0000	0.2617	0.0000	0.1472	0.3363
	4.0000	0.0767	0.1090	0.0000	0.1636	0.0000	0.6507
n = 10							
	1.0000	0.7902	0.0327	0.0000	0.0477	0.0000	0.1294
	2.0000	0.5934	0.0000	0.0857	0.0000	0.0477	0.2732
	3.0000	0.3748	0.0589	0.0000	0.0857	0.0000	0.4807
	4.0000	0.1972	0.0000	0.0589	0.0000	0.0327	0.7111
n = 50							
	1.0000	0.8398	0.0000	0.0000	0.0000	0.0000	0.1602
	2.0000	0.6618	0.0000	0.0000	0.0000	0.0000	0.3382
	3.0000	0.4640	0.0000	0.0000	0.0000	0.0000	0.5360
	4.0000	0.2442	0.0000	0.0000	0.0000	0.0000	0.7558
n = 500							
	1.0000	0.8398	0.0000	0.0000	0.0000	0.0000	0.1602
	2.0000	0.6618	0.0000	0.0000	0.0000	0.0000	0.3382
	3.0000	0.4640	0.0000	0.0000	0.0000	0.0000	0.5360
	4.0000	0.2442	0.0000	0.0000	0.0000	0.0000	0.7558

Section 1-2

Gambler's Ruin with Catastrophes in Finite Time

In this section, a generalization of the original gambler's ruin problem is considered. An added probability, r, of a catastrophe is assumed. That is, a chance that at any point during the betting a player can lose all of his money is now possible. A computer system may be modeled in a similar manner. A hard drive may be thought of as adding bytes of information with probability p and losing or processing information with probability q, but at any point in time the hard drive might crash losing all of its information. The lattice path counting result found in Proposition 1-1.3, some algebra, and basic concepts of probability, provide the tools required to solve the gambler's ruin problem with catastrophes. In this section, we present two different methods of solution. The second method appears more general, allowing us to consider more general related problems (see Section 3). However, in this present section, our primary goal is to obtain a general analytic expression for the probability of being ruined (in finite time) when the gambler's ruin problem is augmented to include catastrophe probabilities.

Motivated by queueing and population models, consider that at any step k=1,2,..(H-1) there is a probability r of a catastrophe taking you to state zero as shown in Figure 5.

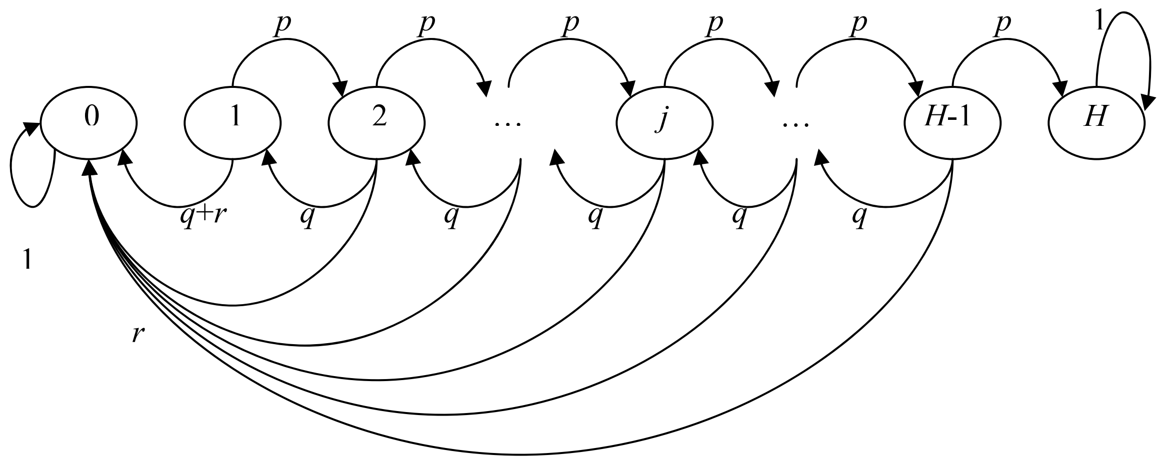

Figure 5 - Gambler's Ruin with Catastrophes

where $p+q+r=1$, $0 \le p,q,r<1$, and $1 \le j \le H-1$. With transition matrix

$$P=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ q+r & 0 & p & 0 & 0 & & 0 \\ r & q & 0 & p & 0 & & 0 \\ r & 0 & q & 0 & p & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ r & & & & q & 0 & p \\ 0 & & & & & 0 & 1 \end{bmatrix}$$

then P^n has entries $P_{j,k}^{(n)}$.

Note: $P_{j,0}^{(n)}$ and $P_{j,H}^{(n)}$ are the probabilities of being absorbed by time *n*. As before, we wish to find an explicit representation for $P_{j,k}^{(n)}$. Assume $1 \le j,k \le H-1$ as shown in Figure 6.

Again, let

$P_{j,k}^{(n)}$ = Prob(being in state *k* in *n*-steps | initially at state *j*).

$R_{j,0}^{(n)}$ = Prob(absorption at state zero in *n*-steps | initially at state *j*).

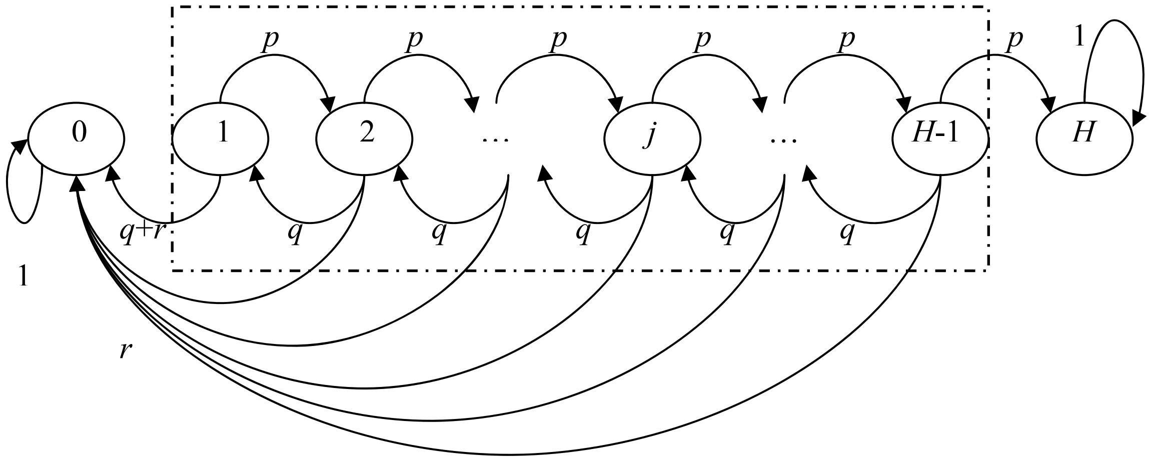

Figure 6 - Gambler's Ruin with Catastrophes $P_{j,k}^{(n)}$ in the box

The following corollary follows from Corollary 1-1.4

Corollary 1-2.1

Case 1 $1 \le j, k \le H-1$

$$P_{j,k}^{(n)} = \frac{2}{H} p^{\frac{k-j}{2}} q^{\frac{j-k}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi k}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{n}$$

Case 2 $1 \le j \le H-1$

$$P_{j,H}^{(n)} = \frac{2}{H} q^{1-H+j} p^{1+H-j} \sum_{u=1}^{H} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right) - 1\right]^{-1} \sin\left(\frac{u\pi(H-j)}{H}\right) \sin\left(\frac{u\pi}{H}\right) \cdot \left[\left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^{n} - 1\right]$$

Proof.

In Case 1, the number of sample paths inside the box of Figure 6 is the same as the number of sample paths inside the box of Figure 2. The only difference is that now

$p+q<1$ since $p+q+r=1$. Therefore the same counting formula (Proposition 1-1.3) holds and Case 1 follows here as it did in Corollary 1-1.4. For Case 2, finding $P_{j,H}^{(n)}$ is equivalent to the problem of finding $P_{H-j,0}^{(n)}$ with the p's and q's reversed in Figure 2. In Section 1, the probability of moving from state j to state zero in n-steps was shown in Corollary 1-1.4 to be

$$P_{j,0}^{(n)} = \frac{2}{\mathrm{H}} p^{\frac{1-j}{2}} q^{\frac{1+j}{2}} \sum_{u=1}^{H} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right) - 1 \right]^{-1} \sin\left(\frac{u\pi j}{H}\right) \sin\left(\frac{u\pi}{H}\right) \left[\left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right) \right)^{n} - 1 \right]$$

Switching the p's and the q's and replacing the initial state j with H-j gives,

$$P_{j,H}^{(n)} = \frac{2}{H} q^{1-H+j} p^{1+H-j} \sum_{u=1}^{H} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right) - 1 \right]^{-1} \sin\left(\frac{u\pi(H-j)}{H}\right) \sin\left(\frac{u\pi}{H}\right) \left[\left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right) \right)^{n} - 1 \right]$$

which establishes Case 2 in Corollary 1-2.1.

To determine the ruin probabilities on Figure 5 we use two different methods.

Theorem 1-2.2 For $1 \le j \le H-1$

$$R_{j,0}^{(n)} = 1 - \left[\sum_{l=0}^{H-1} \left\{ \frac{2}{H} q^{\frac{n-k+j}{2}} p^{\frac{n+l-j}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi l}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(2\cos\left(\frac{u\pi}{H}\right) \right)^{n} \right\} \right.$$
$$\left. + \frac{2}{H} q^{1-H+j} p^{1+H-j} \sum_{u=1}^{H} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right) - 1 \right]^{-1} \sin\left(\frac{u\pi(H-j)}{H}\right) \sin\left(\frac{u\pi}{H}\right) \left[\left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right) \right)^{n} - 1 \right] \right]$$

Proof.

Since $p + q + r = 1$,

$$R_{j,0}^{(n)} = 1 - \left[\sum_{l=0}^{H-1} P_{j,l}^{(n)} + P_{j,H}^{(n)} \right]$$

holds, and the ruin probability follows from Corollary 1-2.1.

Note: the preceding theorem depends upon the ruin probability r in an implicit way since $p + q + r = 1$. The following theorem gives a more general alternative expression for the ruin probabilities on Figure 5. Here the dependency upon r is explicit.

Theorem 1-2.3 For $1 \le j \le H-1$

$$R_{j,0}^{(n)} = \sum_{l=1}^{n-1} \frac{2}{H} q^{\frac{l-1+j}{2}} p^{\frac{l+1-j}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l} \cdot q$$
$$+ \sum_{l=1}^{n-1} \sum_{k=1}^{H-1} \frac{2}{H} q^{\frac{l-k+j}{2}} p^{\frac{l+k-j}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi k}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l} \cdot r$$

Proof

The theorem will be shown assuming $p + q + r \le 1$. The argument proceeds by looking at where you were at one step back from ruin, assuming you are ruined in l steps. This means that we either started at state j and went to state 1 in l-1 steps and then on the last step went to state 0 with probability q, a loss; or starting at j we journeyed to state k in l-1 steps and then on the last step went to state 0 with probability r, a catastrophe. If we add up over all possible paths having l steps and over all possible last positions k, we obtain

$$R_{j,0}^{(n)} = \sum_{l=1}^{n-1} P_{j,1}^{(l)} \cdot q + \sum_{l=1}^{n-1} \sum_{k=1}^{H-1} P_{j,k}^{(l)} \cdot r$$

Theorem 1-2.3 now follows from substitution of Case 1 of Corollary 1-2.1.

Note: This path counting argument applies to the more general situation of $p + q + r \le 1$.

Example with Catastrophes

A player, playing at an illegal casino, starts with a given amount of money, j= 1,2,3 or 4 dollars, and makes a series of one dollar bets placing a bet. Assume the player either wins a dollar or loses a dollar on each bet. At any time during the player's betting the casino has probability .125 of being closed down, causing the player to lose all of his money.

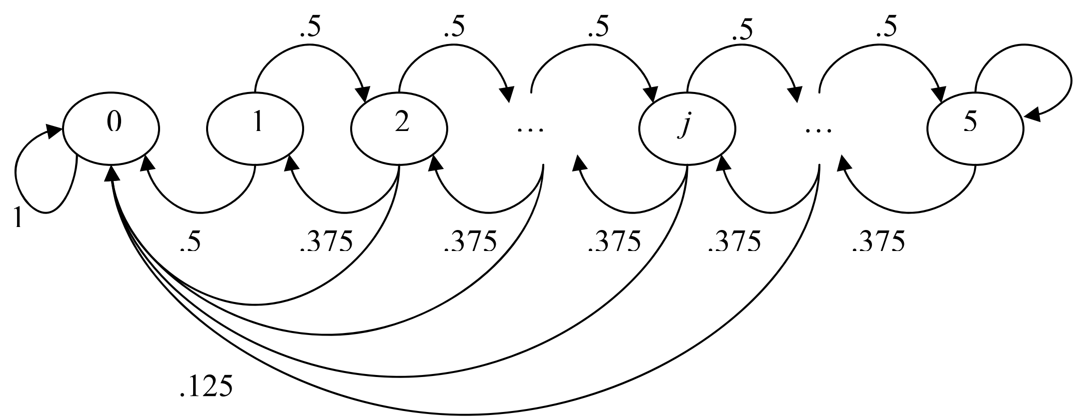

Figure 7

H = 5, p=1/2 (probability of a win), q=3/8 (probability of a loss), r=1/8 (probability of catastrophe), j = starting amount

$P_{j,k}^{(n)}$ have been calculated in the following table, listed as P(j,k). For example the probability for being in state three after ten bets given you stated with one dollar is found in the n = 10 table, by picking off the forth row, P(j,2) column, entry. $P_{1,3}^{(10)}$ =.0170

	j	P(j,5)	P(j,1)	P(j,2)	P(j,3)	P(j,4)	P(j,0)
n = 5							
	1.0000	0.0198	0.0000	0.0879	0.0000	0.0703	0.8220
	2.0000	0.0824	0.0659	0.0000	0.1406	0.0000	0.7111
	3.0000	0.1934	0.0000	0.1055	0.0000	0.0879	0.6133
	4.0000	0.4717	0.0297	0.0000	0.0659	0.0000	0.4327
n = 10							
	1.0000	0.0392	0.0079	0.0000	0.0170	0.0000	0.9359
	2.0000	0.1045	0.0000	0.0206	0.0000	0.0170	0.8579
	3.0000	0.2360	0.0096	0.0000	0.0206	0.0000	0.7338
	4.0000	0.4901	0.0000	0.0096	0.0000	0.0079	0.4925
n = 20							
	1.0000	0.0418	0.0002	0.0000	0.0005	0.0000	0.9575
	2.0000	0.1114	0.0000	0.0006	0.0000	0.0005	0.8876
	3.0000	0.2416	0.0003	0.0000	0.0006	0.0000	0.7576
	4.0000	0.4957	0.0000	0.0003	0.0000	0.0002	0.5038
n = 30							
	1.0000	0.0418	0.0000	0.0000	0.0000	0.0000	0.9581
	2.0000	0.1116	0.0000	0.0000	0.0000	0.0000	0.8884
	3.0000	0.2417	0.0000	0.0000	0.0000	0.0000	0.7582
	4.0000	0.4959	0.0000	0.0000	0.0000	0.0000	0.5041
n = 50							
	1.0000	0.0418	0.0000	0.0000	0.0000	0.0000	0.9582
	2.0000	0.1116	0.0000	0.0000	0.0000	0.0000	0.8884
	3.0000	0.2417	0.0000	0.0000	0.0000	0.0000	0.7583
	4.0000	0.4959	0.0000	0.0000	0.0000	0.0000	0.5041

The numbers on this table are round to four significant digits.

Summarizing the last corollary and the two past theorems, the n-step transition probability functions of the gambler's ruin problem with catastrophes (Figure 5) starting at state j where $1 \le j \le H-1$ are:

$$P_{j,k}^{(n)} = \frac{2}{H} q^{\frac{n-k+j}{2}} p^{\frac{n+k-j}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi k}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(2\cos\left(\frac{u\pi}{H}\right)\right)^{n}$$

$$P_{j,H}^{(n)} = \frac{2}{H} q^{1-H+j} p^{1+H-j} \sum_{u=1}^{H} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)-1\right]^{-1} \sin\left(\frac{u\pi(H-j)}{H}\right) \sin\left(\frac{u\pi}{H}\right) \left[\left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^{n} - 1\right]$$

$$\begin{aligned} R_{j,0}^{(n)} = 1 - \Bigg[& \sum_{l=0}^{H-1} \left\{ \frac{2}{H} q^{\frac{n-k+j}{2}} p^{\frac{n+l-j}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi l}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(2\cos\left(\frac{u\pi}{H}\right)\right)^{n} \right\} \\ & + \frac{2}{H} q^{1-H+j} p^{1+H-j} \sum_{u=1}^{H} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)-1\right]^{-1} \sin\left(\frac{u\pi(H-j)}{H}\right) \sin\left(\frac{u\pi}{H}\right) \left[\left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^{n} - 1\right] \Bigg] \end{aligned}$$

Alternatively,

$$\begin{aligned} R_{j,0}^{(n)} = & \sum_{l=1}^{n-1} \frac{2}{H} q^{\frac{l-1+j}{2}} p^{\frac{l+1-j}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l} \cdot q \\ & + \sum_{l=1}^{n-1} \sum_{k=1}^{H-1} \frac{2}{H} q^{\frac{l-k+j}{2}} p^{\frac{l+k-j}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi k}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l} \cdot r \end{aligned}$$

whenever $p + q + r \le 1$.

Section 1-3

Gambler's Ruin with Catastrophes and Windfalls in Finite Time

In this section, we generalize the methods of Section 2 to determine the ruin probabilities of a gambler's ruin system having both catastrophes and windfalls. The method discussed in Theorem 1-2.3 applies directly to give an explicit, analytic expression for the ruin probabilities of a system having catastrophes and windfalls in addition to the usual win/lose transitions. As before, our approach is built upon the path counting result of Proposition 1-1.3 in Section 1. The main idea is that an eventual catastrophe or windfall, when it occurs, will only happen at the end of a journey. This allows us to consider walks on Figure 8 below to be essentially lattice paths until the end of the walk. Proposition 1-1.3 tells us how to count such paths. Consider

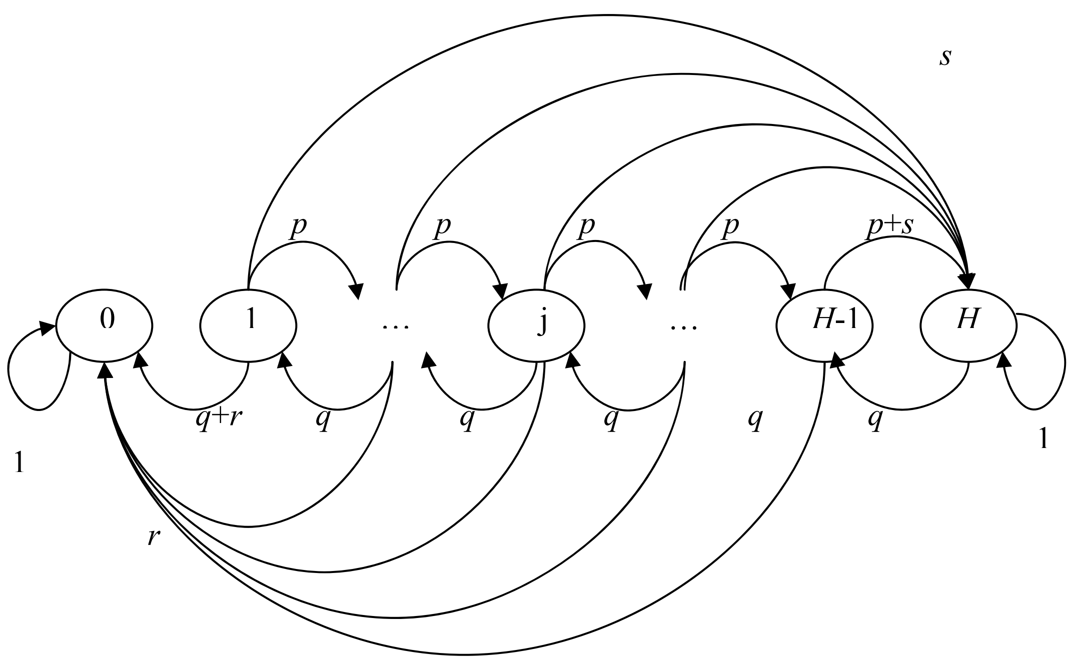

Figure 8 - Gambler's Ruin with Catastrophes and Windfalls

with transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ q+r & 0 & p & 0 & 0 & & s \\ r & q & 0 & p & 0 & & s \\ r & 0 & q & 0 & p & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & s \\ r & & & & q & 0 & p+s \\ 0 & & & & & 0 & 1 \end{bmatrix}$$

then P^n has entries $P_{j,k}^{(n)}$. Note: $P_{j,0}^{(n)}$ and $P_{j,H}^{(n)}$ are the probabilities of being absorbed by time *n*. Consider that at any step *k*=1,2 , … ,*H*-1 there is a probability *r* of a catastrophe taking you to state zero and that there is probability *s* of a windfall taking you to state *H* with $p+q+r+s=1$, $0 \le p,q,r \le 1$, and $1 \le j \le H-1$. As before we wish to find an explicit representation for $P_{j,k}^{(n)}$.

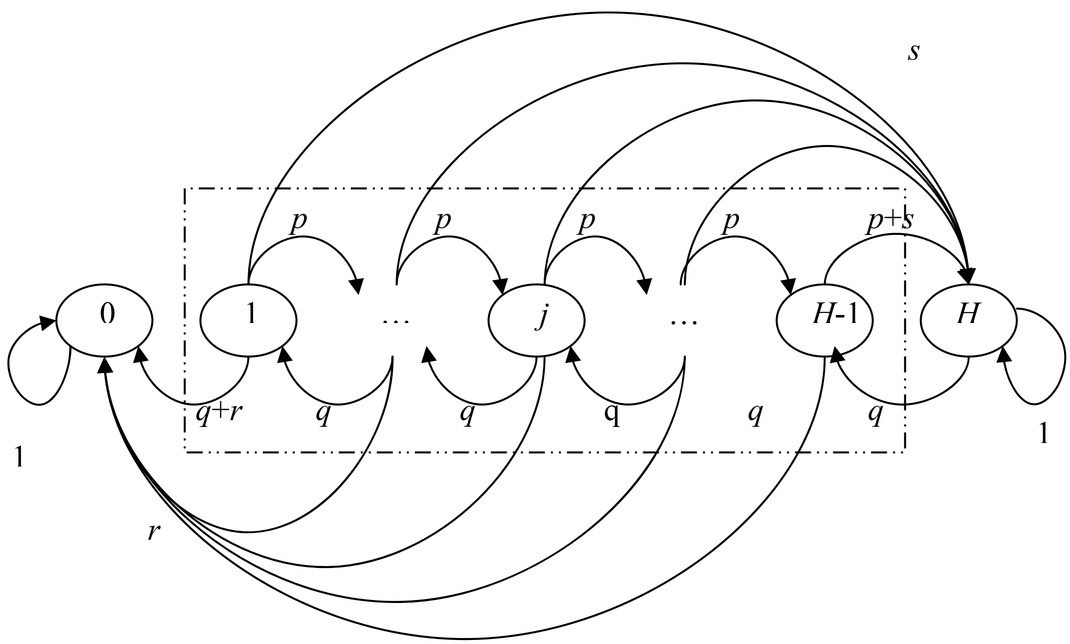

Figure 9 Gambler's Ruin with Catastrophes $P_{j,k}^{(n)}$ in the box

The transition probability functions are determined for two distinct cases by the following corollary.

Corollary 1-3.1

Case 1 For $1 \le j, k \le H - 1$

$$P_{j,k}^{(n)} = \frac{2}{H} p^{\frac{k-j}{2}} q^{\frac{j-k}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi k}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{n}$$

Case 2 For $1 \le j \le H - 1$

$$R_{j,0}^{(n)} = \sum_{l=1}^{n-1} \frac{2}{H} q^{\frac{l-1+j}{2}} p^{\frac{l+1-j}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l} \cdot q$$
$$+ \sum_{l=1}^{n-1} \sum_{k=1}^{H-1} \frac{2}{H} q^{\frac{l-k+j}{2}} p^{\frac{l+k-j}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi k}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l} \cdot r$$

Proof.

For Case 1, the proof follows as before from Proposition 1-1.3. Case 2 follows from the previous path counting result discussed in Theorem 1-2.3.

□

In a similar way, this path counting technique allows a direct approach for finding the probability of reaching your goal, $P_{j,H}^{(n)}$, in *n* steps. This time

$$P_{j,H}^{(n)} = \sum_{l=1}^{n-1} P_{j,H-1}^{(l)} \cdot p + \sum_{l=1}^{n-1} \sum_{k=1}^{H-1} P_{j,k}^{(l)} \cdot s$$

which gives

$$P_{j,H}^{(n)}=\sum_{l=1}^{n-1}\frac{2}{H}q^{\frac{l-H+1+j}{2}}p^{\frac{l+1-j}{2}}\sum_{u=1}^{H}\sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi j}{H}\right)\left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l}\cdot p$$
$$+\sum_{l=1}^{n-1}\sum_{k=1}^{H-1}\frac{2}{H}q^{\frac{l-k+j}{2}}p^{\frac{l+k-j}{2}}\sum_{u=1}^{H}\sin\left(\frac{u\pi k}{H}\right)\sin\left(\frac{u\pi j}{H}\right)\left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l}\cdot s \, .$$

In summary, the general *n*-step transition probability functions of the gambler's ruin with catastrophes and windfalls are, starting from state j, $1\le j\le H-1$, to k, $1\le k\le H-1$,

$$P_{j,k}^{(n)}=\frac{2}{H}q^{\frac{n-k+j}{2}}p^{\frac{n+k-j}{2}}\sum_{u=1}^{H}\sin\left(\frac{u\pi k}{H}\right)\sin\left(\frac{u\pi j}{H}\right)\left(2\cos\left(\frac{u\pi}{H}\right)\right)^{n}.$$

The ruin probability, starting from state j, $1\le j\le H-1$, is

$$R_{j,0}^{(n)}=\sum_{l=1}^{n-1}\frac{2}{H}q^{\frac{l-1+j}{2}}p^{\frac{l+1-j}{2}}\sum_{u=1}^{H}\sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi j}{H}\right)\left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l}\cdot q$$
$$+\sum_{l=1}^{n-1}\sum_{k=1}^{H-1}\frac{2}{H}q^{\frac{l-k+j}{2}}p^{\frac{l+k-j}{2}}\sum_{u=1}^{H}\sin\left(\frac{u\pi k}{H}\right)\sin\left(\frac{u\pi j}{H}\right)\left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l}\cdot r$$

The goal probability, starting from state j, $1\le j\le H-1$, is

$$P_{j,H}^{(n)}=\sum_{l=1}^{n-1}\frac{2}{H}q^{\frac{l-H+1+j}{2}}p^{\frac{l+1-j}{2}}\sum_{u=1}^{H}\sin\left(\frac{u\pi}{H}\right)\sin\left(\frac{u\pi j}{H}\right)\left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l}\cdot p$$
$$+\sum_{l=1}^{n-1}\sum_{k=1}^{H-1}\frac{2}{H}q^{\frac{l-k+j}{2}}p^{\frac{l+k-j}{2}}\sum_{u=1}^{H}\sin\left(\frac{u\pi k}{H}\right)\sin\left(\frac{u\pi j}{H}\right)\left(2\cos\left(\frac{u\pi}{H}\right)\right)^{l}\cdot s$$

As far as we know, these formulae are new in the catastrophe and windfall setting.

Chapter 2

Gambler's Ruin in Infinite Time

Introduction

In this chapter, we again consider the classical gambler's ruin problem but with an infinite amount of time. More generally, we reconsider each of the three models presented in Chapter 1 under the assumption of having an indefinite amount of time until absorption. This time, our techniques do not explicitly use path counting. The problem is addressed as solving a system of recurrence relations under appropriate boundary conditions. Two techniques are employed: difference equations and probability generating functions. In particular, the infinite time gambler's ruin with catastrophes problem is solved in Section 2 using the theory of second order, constant coefficient recurrence relations and some calculus and probability theory to hold it all together. This result and the method are then generalized in Section 3 to solve the gambler's ruin with both catastrophes and windfalls in infinite time using difference equation techniques. An interesting probability generating function approach may also be used to determine the ruin probabilities in the catastrophe and windfall case. This solution method is also presented in Section 3.

Section 2-1

The Gambler's Ruin Problem

We reconsider the classical gambler's ruin Problem but now with an infinite amount of time,

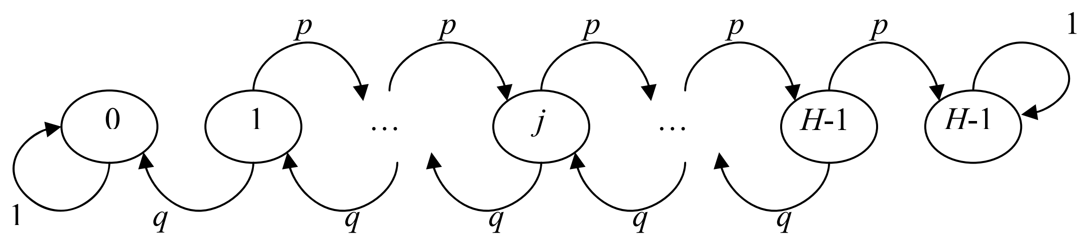

Figure 10 - The state diagram for the gambler's ruin problem

where $0 < p, q < 1$, $p + q = 1$ and $1 \le j \le H - 1$. The ruin probability is the chance of eventually reaching state "0" before state "*H*". The classical gambler's ruin problem in infinite time has the following elegant solution, see [4, 11].

$$R_j = P_j(T_0 < T_H) = \left\{ \begin{array}{ll} \dfrac{\left(\dfrac{q}{p}\right)^j - \left(\dfrac{q}{p}\right)^H}{1 - \left(\dfrac{q}{p}\right)^H} & for \quad p \neq q \\ \\ 1 - \dfrac{j}{H} & for \quad p = q \end{array} \right\}$$

The problem has been solved in many ways ([11]).

Section 2-2

Gambler's Ruin with Catastrophes in Infinite Time

This section looks at the same set-up as in Section 1-2 but with infinite time. We are looking for the ruin probabilities, that is, probabilities of eventual absorption at state zero. In this section, we derive expressions for both the ruin probabilities and the probability of reaching our goal, H, given that you start at state j. These are obtained using the theory of linear, constant coefficient recurrence relations.

For infinite time, recall this Markov chain was first described in Section 2 of Chapter 1.

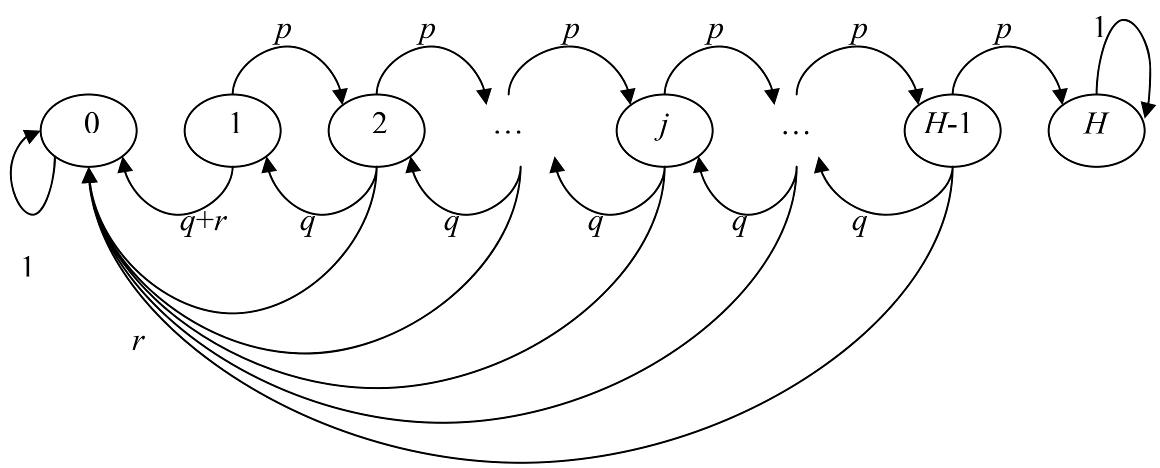

Figure 11 - Gambler's Ruin with Catastrophes

Let

$$R_j = \text{Prob(eventual absorption at state zero | initially at state } j)$$

and

$$P_j = \text{Prob(eventual absorption at state } H \text{ | initially at state } j).$$

The main result of this section is given in the following theorem which is reproduced here from some joint work appearing in [5].

Theorem 2-2.1 For $1 \le j \le H-1$

$$R_j = 1 - \left[\frac{(2p)^{H-j} \cdot \sum_{u=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \binom{H}{2u+1} (1-4pq)^u}{\sum_{u=0}^{\left\lfloor \frac{H-1}{2} \right\rfloor} \binom{H}{2u+1} (1-4pq)^u} \right]$$

Proof .

Conditioning on the first step,

$$\begin{aligned}
&P_0 = 0 \\
&P_1 = (q+r)P_0 + pP_2 = qP_0 + pP_2 \\
&P_2 = rP_0 + qP_1 + pP_3 = qP_1 + pP_3 \\
&\vdots \\
&P_i = qP_{i-1} + pP_{i+1} \qquad for \qquad i = 1,2,\ldots,H-1 \\
&\vdots \\
&P_{H-1} = qP_{H-2} + pP_H \\
&P_H = 1
\end{aligned}$$

If you start in state *k* the chance of being ruined can be found by seeing where you are after one step. After one step you could be at state *k*+1 with probability *p* then get ruined, at *k*-1 with probability *q* then get ruined or at state zero with probability *r* then get ruined. The probability after one step of being absorbed at *H* given that you began at state zero is zero since zero is an absorbing state. The equations form a system of second order, linear, constant-coefficient difference equations.

Thus for $k = 1,2,3,...,H-1$,

$$pP_{k+1} - P_k + qP_{k-1} = 0$$

with characteristic equation,

$$px^2 - x + q = 0 .$$

The roots of this equation are $\frac{1 \pm \sqrt{1-4pq}}{2p}$. Note that these roots are distinct since *r*

is positive. Hence the general solution of the recurrence is of the form (see [8])

$$P_k = c_1\left[\frac{1+\sqrt{1-4pq}}{2p}\right]^k + c_2\left[\frac{1-\sqrt{1-4pq}}{2p}\right]^k$$

where c_1 and c_2 are constants to be determined by the two boundary conditions

$P_0 = 0$ and $P_H = 1$. This gives

$$0 = c_1 + c_2$$

$$1 = c_1\left[\frac{1+\sqrt{1-4pq}}{2p}\right]^H + c_2\left[\frac{1-\sqrt{1-4pq}}{2p}\right]^H$$

with solution

$$c_2 = -c_1$$

$$c_1 = \frac{(2p)^H}{\left(1+\sqrt{1-4pq}\right)^H - \left(1-\sqrt{1-4pq}\right)^H}$$

Thus,

$$P_k = \frac{(2p)^H}{\left(1+\sqrt{1-4pq}\right)^H - \left(1-\sqrt{1-4pq}\right)^H} \cdot \left\{\left[\frac{1+\sqrt{1-4pq}}{2p}\right]^k - \left[\frac{1-\sqrt{1-4pq}}{2p}\right]^k\right\}$$

$$P_k = \frac{(2p)^{H-k} \cdot \left[\left(1+\sqrt{1-4pq}\right)^k - \left(1-\sqrt{1-4pq}\right)^k\right]}{\left(1+\sqrt{1-4pq}\right)^H - \left(1-\sqrt{1-4pq}\right)^H}$$

$$P_k = \left[\frac{(2p)^{H-k} \cdot \sum_{u=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{H}{2u+1} (1-4pq)^u}{\sum_{u=0}^{\left\lfloor \frac{H-1}{2} \right\rfloor} \binom{H}{2u+1} (1-4pq)^u}\right]$$

So,

$$R_k = 1 - P_k,$$

completing the proof.

Section 2-3

Gambler's Ruin with Catastrophes and Windfalls in Infinite Time

In this section, we consider a system of linear constant coefficient recurrence relations for the ruin probabilities of Figure 12 given that you start at state j. We present two methods of solution. The first approach uses probability generating functions. Our second solution method (which we gratefully acknowledge was suggested by thesis committee member, Dr. Randall J. Swift) is a difference equation approach which essentially generalizes the arguments presented in the previous section.

Reconsider the Markov chain illustrated in Figure 12.

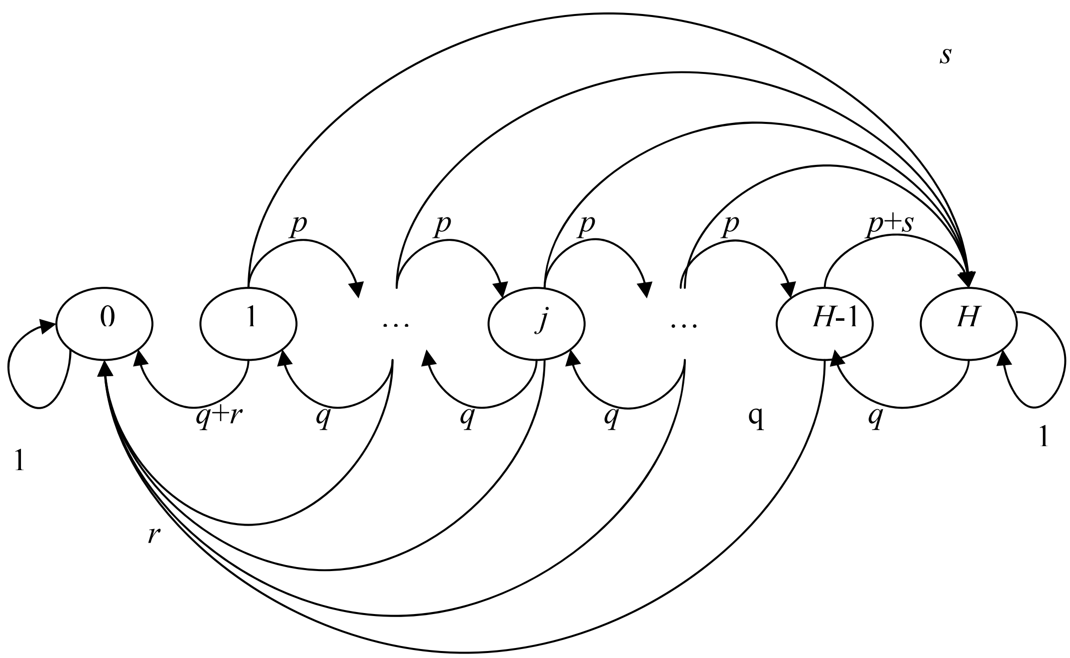

Figure 12 - Gambler's Ruin with Catastrophes and Windfalls

We seek the probability of eventually being ruined, R_j = Prob(eventual absorption at state zero | initially at state j).

Theorem 2-3.1 For the Markov chain described by Figure 12, the R_j are given below.

$$R_0 = 1$$

$$R_1 = \frac{\alpha\rho_1\rho_2^{\,2} + \beta\rho_1^{\,2}\rho_2 - \alpha p\rho_2^{\,2} - \beta p\rho_1^{\,2}}{q\rho_1^{\,2}\rho_2^{\,2} + \alpha p\rho_1\rho_2^{\,2} + \beta p\rho_1^{\,2}\rho_2}$$

$$R_j = \sum_{k=0}^{j} a_k b_{j-k}$$

where

$$a_0 = \frac{p}{q}$$

$$a_1 = \frac{p}{q}R_1 - \frac{1}{q}$$

$$a_k = \frac{r}{q} \quad for \quad k = 2,3,\ldots,H, \text{ and}$$

$$b_i = \left(\frac{\alpha}{\rho_1^{\,i+1}} + \frac{\beta}{\rho_2^{\,i+1}}\right) \quad for \quad i = 0,1,2,\ldots,H$$

where

$$\rho_1 = \frac{1+\sqrt{1-4pq}}{2q} \qquad \rho_2 = \frac{1-\sqrt{1-4pq}}{2q}$$

$$\alpha = \frac{1}{(\rho_2 - \rho_1)} \quad \text{and} \quad \beta = \frac{1}{(\rho_1 - \rho_2)}$$

Proof.

Conditioning on the first step gives,

$$\begin{aligned}
R_0 &= 1 \\
R_1 &= qR_0 + pR_2 + r \\
R_2 &= qR_1 + pR_3 + r \\
&\vdots \\
R_i &= qR_{i-1} + pR_{i+1} + r \qquad i = 1,2,...,H-1 \\
&\vdots \\
R_{H-1} &= qR_{H-2} + pR_H + r \\
R_H &= 0
\end{aligned}$$

where $p+q+r+s=1$. So

$$R_i = qR_{i-1} + pR_{i+1} + r$$

for $i = 1,2,...,H-1$ with initial conditions, $R_0 = 1$ and $R_H = 0$.

$$pR_{n+1} - R_n + qR_{n-1} = -r$$

The ruin probabilities, R_i ,can be found using the technique of generating functions (see page 10 of [13]).

Let

$$R(x) = \sum_{i=0}^{H} R_i x^i \qquad \text{and} \qquad D(x) = \sum_{i=1}^{H-1} r x^i$$

Beginning with

$$pR_{n+1} - R_n + qR_{n-1} = -r$$

and multiplying through by x^n gives,

$$pR_{n+1}x^n - R_n x^n + qR_{n-1}x^n = -rx^n$$

$$p\sum_{n=1}^{H-1} R_{n+1}x^n - \sum_{n=1}^{H-1} R_n x^n + q\sum_{n=1}^{H-1} R_{n-1}x^n = -r\sum_{n=1}^{H-1} x^n$$

$$\frac{p\sum_{n=1}^{H-1} R_{n+1}x^{n+1}}{x} - \left[R(x) - 1\right] + qx\sum_{n=1}^{H-1} R_{n-1}x^{n-1} = -rx\sum_{n=1}^{H-1} x^{n-1}$$

$$\frac{p\left[R(x) - 1 - R_1 x\right]}{x} - \left[R(x) - 1\right] + qx\left[R(x) - R_{H-1}x^{H-1}\right] = \frac{-rx\left(1 - x^{H-1}\right)}{\left(1 - x\right)}$$

Multiplying through by $x(1-x)$,

$$\begin{aligned} &p\left[R(x) - 1 - R_1 x\right](1-x) - x(1-x)\left[R(x) - 1\right] + qx^2(1-x)\left(R(x) - R_{H-1}x^{H-1}\right) \\ &\quad = -rx^2\left(1 - x^{H-1}\right) \end{aligned}$$

$$\begin{aligned} &p(1-x)R(x) - x(1-x)R(x) + qx^2(1-x)R(x) \\ &\quad = p(1-x)(1+R_1 x) + x(x-1) - rx^2\left(1 - x^{H-1}\right) + qR_{H-1}x^{H+1}(1-x) \end{aligned}$$

Solving for $R(x)$,

$$\begin{aligned} &R(x)\left[p(1-x) - x(1-x) + qx^2(1-x)\right] \\ &\quad = p(1-x)(1+R_1 x) + x(x-1) - rx^2\left(1 - x^{H-1}\right) + qR_{H-1}x^{H+1}(1-x) \end{aligned}$$

$$\begin{aligned} &R(x)(1-x)\left[p - x + qx^2\right] \\ &\quad = -rx^2\left(1 - x^{H-1}\right) + p(1-x)(1+R_1 x) + x(x-1) + qR_{H-1}x^{H+1}(1-x) \end{aligned}$$

Dividing through by $(1-x)$,

$$R(x)(p - x + qx^2) = \frac{-rx^2\left(1 - x^{H-1}\right)}{(1-x)} + p(1+R_1 x) - x + qR_{H-1}x^{H+1}$$

Factoring out a q,

$$qR(x)\left(x^2 - \frac{1}{q}x + \frac{p}{q}\right) = \frac{-rx^2\left(1 - x^{H-1}\right)}{(1-x)} + p(1+R_1 x) - x + qR_{H-1}x^{H+1}$$

The roots of the quadratic on the left hand side of the preceding equation are,

$$\rho_1, \rho_2 = \frac{\frac{1}{q} \pm \sqrt{\left(\frac{1}{q}\right)^2 - 4\frac{p}{q}}}{2} = \frac{\frac{1}{q} \pm \sqrt{\frac{1}{q^2} - 4\frac{pq}{q^2}}}{2} = \frac{1 \pm \sqrt{1-4pq}}{2q}$$

R_1 and R_{H-1} may now be found. Note that $\rho_1 \neq \rho_2$ assuming r>0 or s>0. We have

$$qR(x)(x-\rho_1)(x-\rho_2) = \frac{-rx^2\left(1-x^{H-1}\right)}{(1-x)} + p(1+R_1x) - x + qR_{H-1}x^{H+1}$$

Recall the formula for the finite sum of a geometric series

$$\frac{1-x^{H-1}}{1-x} = 1 + x + x^2 + \ldots + x^{H-2}$$

So,

$$qR(x)(x-\rho_1)(x-\rho_2) = -rx^2\left[1 + x + x^2 + \ldots + x^{H-2}\right] + p(1+R_1x) - x + qR_{H-1}x^{H+1}$$

Dividing through by $q(x-\rho_1)(x-\rho_2)$ gives

$$R(x) = \frac{\left[\frac{-r}{q}x^2\left[1 + x + x^2 + \ldots + x^{H-2}\right] + \frac{p}{q}(1+R_1x) - \frac{x}{q} + R_{H-1}x^{H+1}\right]}{(x-\rho_1)(x-\rho_2)}$$

By partial fractions,

$$\frac{1}{(x-\rho_1)(x-\rho_2)} = \frac{A}{(x-\rho_1)} + \frac{B}{(x-\rho_2)}$$

$$\text{where} \quad B = \frac{1}{(\rho_2-\rho_1)} \quad A = \frac{1}{(\rho_1-\rho_2)}$$

Now $\frac{A}{(x-\rho_1)}$ and $\frac{B}{(x-\rho_2)}$ can be represented as power series.

$$\frac{A}{(x-\rho_1)}=\frac{A}{\rho_1\left(\frac{x}{\rho_1}-1\right)}=\frac{A}{-\rho_1}\frac{1}{1-\frac{x}{\rho_1}}=\frac{A}{-\rho_1}\left[1+\frac{x}{\rho_1}+\frac{x^2}{\rho_1^{\;2}}+\frac{x^3}{\rho_1^{\;3}}+\ldots\right]$$

$$\frac{B}{(x-\rho_2)}=\frac{B}{-\rho_2}\left[1+\frac{x}{\rho_2}+\frac{x^2}{\rho_2^{\;2}}+\frac{x^3}{\rho_2^{\;3}}+\ldots\right]$$

Then

$$R(x)=\left[\frac{-r}{q}x^2\left[1+x+x^2+\ldots+x^{H-2}\right]+\frac{p}{q}(1+R_1x)-x+R_{H-1}x^{H+1}\right]\cdot\left[\frac{A}{(x-\rho_1)}+\frac{B}{(x-\rho_2)}\right]$$

Replacing $\frac{A}{(x-\rho_1)}+\frac{B}{(x-\rho_2)}$ with the power series gives

$$R(x)=\left[\frac{-r}{q}x^2\left[1+x+x^2+\ldots+x^{H-2}\right]+\frac{p}{q}(1+R_1x)-\frac{x}{q}+R_{H-1}x^{H+1}\right]\cdot\left[\frac{A}{-\rho_1}\left(1+\frac{x}{\rho_1}+\frac{x^2}{\rho_1^{\;2}}+\frac{x^3}{\rho_1^{\;3}}+\ldots\right)+\frac{B}{-\rho_2}\left(1+\frac{x}{\rho_2}+\frac{x^2}{\rho_2^{\;2}}+\frac{x^3}{\rho_2^{\;3}}+\ldots\right)\right]$$

$$R(x)=-\left[\frac{p}{q}+\left(\frac{p}{q}R_1-\frac{1}{q}\right)x-\frac{r}{q}x^2\left[1+x+x^2+\ldots+x^{H-2}\right]+R_{H-1}x^{H+1}\right]\cdot\left[\left(\frac{A}{\rho_1}+\frac{B}{\rho_2}\right)+\left(\frac{A}{\rho_1^{\;2}}+\frac{B}{\rho_2^{\;2}}\right)\cdot x+\left(\frac{A}{\rho_1^{\;3}}+\frac{B}{\rho_2^{\;3}}\right)\cdot x^2+\ldots\right]$$

Note that in general if $c_0+c_1x+c_2x^2+\ldots=\left(a_0+a_1x+a_2x^2+\ldots\right)\left(b_0+b_1x+b_2x^2+\ldots\right)$

then

$$c_0 = a_0 b_0$$
$$c_1 = a_0 b_1 + a_1 b_0$$
$$\vdots$$
$$c_H = a_0 b_H + a_1 b_{H-1} + a_2 b_{H-2} + \ldots + a_H b_0$$

Here

$$a_0 = \frac{p}{q}$$
$$a_1 = \frac{p}{q} R_1 - \frac{1}{q}$$
$$a_i = \frac{r}{q} \quad for \quad i = 2,3,\ldots,H$$

and

$$b_i = \left(\frac{A}{\rho_1{}^i} + \frac{B}{\rho_2{}^i} \right) \quad for \quad i = 0,1,2,\ldots,H$$

with

$$R_1 = \frac{A\rho_1\rho_2{}^2 + B\rho_1{}^2\rho_2 - Ap\rho_2{}^2 - Bp\rho_1{}^2}{q\rho_1{}^2\rho_2{}^2 + Ap\rho_1\rho_2{}^2 + Bp\rho_1{}^2\rho_2}$$

$$\rho_1 = \frac{1+\sqrt{1-4pq}}{2q} \qquad \rho_2 = \frac{1-\sqrt{1-4pq}}{2q}$$

$$A = \frac{1}{(\rho_1 - \rho_2)} \quad and \quad B = \frac{1}{(\rho_2 - \rho_1)}$$

Take $A = \beta \quad and \quad B = \alpha$ to obtain the expression in Theorem 2-3.1. The ruin probabilities, R_j, may now be picked off as the coefficients of the generating function

$$R(x) = \sum_{i=0}^{H} R_i x^i \ .$$

Solving for the ruin probabilities makes it possible to find the goal probabilities, P_i's, using the fact that as time runs off to infinity, $P_i + R_i = 1$ for all *i* since the chance you are at either of the two absorbing states is 1. That is there is 100 percent probability of being ruined or meeting your goal if you allow infinite time. Hence, $P_i = 1 - R_i$ for all $i = 0,1,2,...,H$.

An alternative way to solve the recurrence relation is using a difference equation technique, [2]. Recall the recurrence relation found earlier,

$$R_n = pR_{n+1} + qR_{n-1} + r$$

$$R_{n+1} - \frac{1}{p}R_n + \frac{q}{p}R_{n-1} = -\frac{r}{p}$$

The general solution to the preceding recurrence relation may be found by the sum of the homogeneous solution and a particular solution, [2].

To find a particular solution, assume $R_i = c$, c a constant.

Then

$$c - \frac{1}{p}c + \frac{q}{p}c = -\frac{r}{p}$$

$$c\left(1-\frac{1}{p}+\frac{q}{p}\right)=-\frac{r}{p}$$

$$c=-\frac{r}{p-1+q}$$

But $1=p+q+r+s$, so

$$c=-\frac{r}{-r-s}$$

which gives the particular solution,

$$R_i=c=\frac{r}{r+s}$$

Now $R_{n+1}-\frac{1}{p}R_n+\frac{q}{p}R_{n-1}=-\frac{r}{p}$ has the associated homogeneous equation

$$R_{n+1}-\frac{1}{p}R_n+\frac{q}{p}R_{n-1}=0$$

The associated characteristic equation is

$$x^2-\frac{1}{p}x+\frac{q}{p}=0$$

with roots

$$x_1=\frac{1+\sqrt{1-4pq}}{2p} \text{ and } x_2=\frac{1-\sqrt{1-4pq}}{2p}$$

The general solution may now be written as [2]

$$R_n=c_1{x_1}^n+c_2{x_2}^n+c$$

$$R_n=c_1\left(\frac{1+\sqrt{1-4pq}}{2p}\right)^n+c_2\left(\frac{1-\sqrt{1-4pq}}{2p}\right)^n+\frac{r}{r+s}$$

The constants c_1, c_2 are determined by the initial conditions, $R_0 = 1$ and $R_H = 0$:

$$1 = c_1 + c_2 + \frac{r}{r+s}$$

$$0 = c_1\left(\frac{1+\sqrt{1-4pq}}{2p}\right)^H + c_2\left(\frac{1-\sqrt{1-4pq}}{2p}\right)^H + \frac{r}{r+s}$$

which gives

$$c_1 = \frac{r(2p)^H + s(1-\sqrt{1-4pq})^H}{-(r+s)\left[\left(1+\sqrt{1-4pq}\right)^H - \left(1-\sqrt{1-4pq}\right)^H\right]}$$

$$c_2 = 1 - \frac{r}{r+s} + \frac{r(2p)^H + s(1-\sqrt{1-4pq})^H}{(r+s)\left[\left(1+\sqrt{1-4pq}\right)^H - \left(1-\sqrt{1-4pq}\right)^H\right]}$$

Note that

$$(1+\sqrt{x})^k = \sum_{i=o}^{k}\binom{k}{i}\left(\sqrt{x}\right)^i$$

$$(1-\sqrt{x})^k = \sum_{i=o}^{k}\left(-1\right)^k\binom{k}{i}\left(\sqrt{x}\right)^i$$

$$(1+\sqrt{x})^k - (1-\sqrt{x})^k = 2\sum_{\substack{i=0 \\ i \text{ odd}}}^{k}\binom{k}{i}x^i$$

$$= 2\sqrt{x}\sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor}\binom{k}{2j+1}x^j$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to *x*.

So c_1, c_2 can be rewritten as

$$c_1 = \frac{r(2p)^H + s(1-\sqrt{1-4pq})^H}{-(r+s)\left[2\sqrt{1-4pq}\sum_{j=0}^{\left\lfloor \frac{H-1}{2} \right\rfloor}\binom{H}{2j+1}(1-4pq)^j\right]}$$

and

$$c_2 = 1 - \frac{r}{r+s} + \frac{r(2p)^H + s(1-\sqrt{1-4pq})^H}{(r+s)\left[2\sqrt{1-4pq}\sum_{j=0}^{\left\lfloor \frac{H-1}{2} \right\rfloor}\binom{H}{2j+1}(1-4pq)^j\right]}$$

Thus the ruin probabilities R_k 's are

$$R_n = c_1\left(\frac{1+\sqrt{1-4pq}}{2p}\right)^n + c_2\left(\frac{1-\sqrt{1-4pq}}{2p}\right)^n + \frac{r}{r+s}$$

where

$$c_1 = \frac{r(2p)^H + s(1-\sqrt{1-4pq})^H}{-(r+s)\left[2\sqrt{1-4pq}\sum_{j=0}^{\left\lfloor \frac{H-1}{2} \right\rfloor}\binom{H}{2j+1}(1-4pq)^j\right]}$$

$$c_2 = 1 - \frac{r}{r+s} + \frac{r(2p)^H + s(1-\sqrt{1-4pq})^H}{(r+s)\left[2\sqrt{1-4pq}\sum_{j=0}^{\left\lfloor \frac{H-1}{2} \right\rfloor}\binom{H}{2j+1}(1-4pq)^j\right]}$$

Chapter 3

The General Three State Markov Process

Introduction

In Markov processes, the simplest basic example is the two state Markov process whose transient probability functions are solved in complete detail in most books on the subject, cf. [3, 4]. The next step up in complexity is the three state Markov process. However, to our knowledge, the transient probability functions of this process have not been treated in full generality. If the three state Markov process is addressed at all, only certain special cases are presented (see, for example, the problem section of Chapter 3 in [4]). It turns out that the three state Markov process has three mathematical distinct forms that the transient probability functions may take and that these solutions include trigonometric functions that did not appear in the two state Markov process case. Therefore, for the preceding pedagogically and mathematically interesting reasons, a detailed solution of the transient probability functions for the general three state Markov process is developed here.

Section 3-1

The General Three State Markov Process

In this section the transient probability functions, $P_{i,j}(t)$, of the general three state Markov process are explicitly determined. These transient probability functions are found to have three distinct forms of solution which are fully described. The section concludes by illustrating examples of each solution form.

Consider the general three state Markov process pictured below,

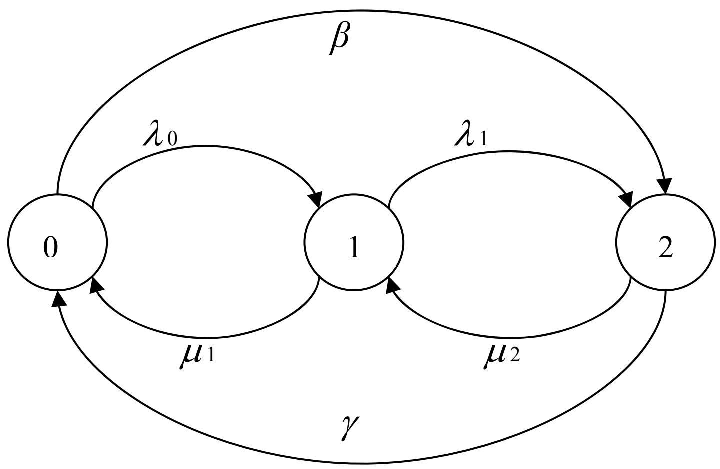

Figure 13 - General three state transition rate diagram

having transition rate matrix

$$Q = \begin{bmatrix} -(\lambda_0 + \beta) & \lambda_0 & \beta \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 \\ \gamma & \mu_2 & -(\gamma + \mu_2) \end{bmatrix}$$

We also denote each entry of Q by $q_{i,j}$, the transition rate of moving from state i to state j. According to the theory of Markov processes,

$$q_{ij} = P'_{ij}(t) \big|_{t=0}$$

for i,j = 0, 1, 2, 3. The matrix Q will also be written as

$$Q = \begin{bmatrix} q_{0,0} & q_{0,1} & q_{0,2} \\ q_{1,0} & q_{1,1} & q_{1,2} \\ q_{2,0} & q_{2,1} & q_{2,2} \end{bmatrix}$$

Moving from state zero to state one, for example occurs at a rate $\lambda_0 = q_{0,1}$, and moving from state zero to state two occurs at rate β = $q_{0,2}$. Since no other transitions out of state zero occur the rate of staying in state zero is the negative sum of the departing rates, [4], that is, $-\left(\lambda_0 + \beta\right) = q_{0,0}$. The rows of Q add up to zero. Diagonal entries are non-positive. Off diagonal terms are non-negative. We are interested in finding the transition probability functions, $P_{i,j}(t)$, where $i, j = 0,1,2$. $P_{i,j}(t)$ is found by solving the Kolmogorov forward or backwards equations,

$$P'(t) = Q \cdot P(t) = P(t) \cdot Q$$

with $P(0) = \mathrm{I}$, where

$$P(t) = \begin{bmatrix} P_{0,0}(t) & P_{0,1}(t) & P_{0,2}(t) \\ P_{1,0}(t) & P_{1,1}(t) & P_{1,2}(t) \\ P_{2,0}(t) & P_{2,1}(t) & P_{2,2}(t) \end{bmatrix}$$

is the matrix of transition probability functions and Q is as above. The solution of the Kolmogorov backwards equation can be written [4] as

$$P(t) = e^{Qt}$$

and the steady state distribution is given by the following distribution.

Lemma 3-1.1 The steady state distribution of the general three state Markov process pictured in Figure 13 is

$$\pi_0 = \frac{\mu_1\mu_2 + \gamma\lambda_1 + \gamma\mu_1}{C}$$

$$\pi_1 = \frac{\lambda_0\mu_2 + \gamma\lambda_0 + \beta\mu_2}{C}$$

$$\pi_2 = \frac{\lambda_0\lambda_1 + \beta\lambda_1 + \beta\mu_1}{C}$$

where

$$C = \beta(\lambda_1 + \mu_1 + \mu_2) + \gamma(\lambda_0 + \lambda_1 + \mu_1) + \lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_2\mu_1$$

Proof

The stationary (and steady state) distribution π may be found see [3] or [4] by solving

$$\pi Q = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix}\begin{bmatrix}-\lambda_0-\beta & \lambda_0 & \beta \\ \mu_1 & -\mu_1-\lambda_1 & \lambda_1 \\ \gamma & \mu_2 & -\gamma-\mu_2\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\end{bmatrix}$$

This gives the following system of equations,

$$(-\lambda_0-\beta)\pi_0+\mu_1\pi_1+\gamma\pi_2=0$$

$$\lambda_0\pi_0+(-\mu_1-\lambda_1)\pi_1+\mu_2\pi_2=0$$

$$\beta\pi_0+\lambda_1\pi_1+(-\gamma-\mu_2)\pi_2=0$$

which may be solved for π_0, π_1, π_2 provided we require π to be a distribution, that is, $\pi_0+\pi_1+\pi_2=1$. Rather than reproducing all the algebraic details to solve this system, since the steady state distribution π is unique it is enough to verify that

$$\pi Q=[0,0,0]$$

when

$$\pi_0=\frac{\mu_1\mu_2+\gamma\lambda_1+\gamma\mu_1}{\beta(\lambda_1+\mu_1+\mu_2)+\gamma(\lambda_0+\lambda_1+\mu_1)+\lambda_0\lambda_1+\lambda_0\mu_2+\mu_2\mu_1}$$

$$\pi_1=\frac{\lambda_0\mu_2+\gamma\lambda_0+\beta\mu_2}{\beta(\lambda_1+\mu_1+\mu_2)+\gamma(\lambda_0+\lambda_1+\mu_1)+\lambda_0\lambda_1+\lambda_0\mu_2+\mu_2\mu_1}$$

$$\pi_2=\frac{\lambda_0\lambda_1+\beta\lambda_1+\beta\mu_1}{\beta(\lambda_1+\mu_1+\mu_2)+\gamma(\lambda_0+\lambda_1+\mu_1)+\lambda_0\lambda_1+\lambda_0\mu_2+\mu_2\mu_1}$$

or

$$\pi_0=\frac{\mu_1\mu_2+\gamma\lambda_1+\gamma\mu_1}{C}\qquad \pi_1=\frac{\lambda_0\mu_2+\gamma\lambda_0+\beta\mu_2}{C}\qquad \pi_2=\frac{\lambda_0\lambda_1+\beta\lambda_1+\beta\mu_1}{C}$$

where

$$C = \beta(\lambda_1 + \mu_1 + \mu_2) + \gamma(\lambda_0 + \lambda_1 + \mu_1) + \lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_2\mu_1$$

This straight forward check is left to the reader and completes the proof.

Since the solutions to $P'(t) = Q \cdot P(t) = P(t) \cdot Q$, are known, [13], to have the form

$$P_{ij}(t) = c_{ij}e^{r_0 t} + a_{ij}e^{r_1 t} + b_{ij}e^{r_2 t}$$

when r_0, r_1, r_2 are distinct eigenvalues of Q with constant coefficients (and solution form $P_{ij}(t) = c_{ij}e^{r_0 t} + a_{ij}e^{r_1 t} + b_{ij}te^{r_1 t}$ when, for example, r_1 is a double root), we now turn our attention to finding the eigenvalues of Q. To determine the eigenvalues of *Q,* recall that the characteristic equation of *Q* is

$$\det[Q - xI] = 0$$

$$\det\begin{bmatrix} -\lambda_0 - \beta - x & \lambda_0 & \beta \\ \mu_1 & -\mu_1 - \lambda_1 - x & \lambda_1 \\ \gamma & \mu_2 & -\gamma - \mu_2 - x \end{bmatrix} = 0$$

Notice that vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of *Q* since $Q \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This follows since each of the rows of *Q* add up to zero. Thus 0 is an eigenvalue of *Q*. This means the characteristic equation of *Q* may be written as

$$x(x^2 + Bx + C) = 0$$

where

$$B = \beta + \gamma + \lambda_0 + \lambda_1 + \mu_1 + \mu_2$$

$$C = \beta(\lambda_1 + \mu_1 + \mu_2) + \gamma(\lambda_0 + \lambda_1 + \mu_1) + \lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_2\mu_1$$

This is the same *C* that appears in Lemma 3-1.1. If r_1, r_2 are the roots of $x^2 + Bx + C = 0$, then 0 and $r_1, r_2 = \dfrac{-B \pm \sqrt{B^2 - 4C}}{2}$ are the eigenvalues of *Q*. There are three distinct cases to consider:

1. $B^2 - 4C > 0$
2. $B^2 - 4C = 0$
3. $B^2 - 4C < 0$

Case 1 $B^2 - 4C > 0$

Since $C > 0$ it follows that $r_1, r_2 \leq 0$ and $r_1 \neq r_2$ since $B^2 - 4C > 0$. In particular, $r_1, r_2, 0$ are three non-positive eigenvalues of *Q*, which are distinct if zero is not a multiple root that is, if $C > 0$. If zero is a multiple root then the Markov process effectively becomes a process having transitions between two or fewer of its three states. Therefore, we henceforth assume $r_1, r_2, 0$ are three non-positive, distinct eigenvalues of *Q*. Then, by [13] for example,

$$P_{ij}(t) = \pi_j + a_{ij}e^{r_1 t} + b_{ij}e^{r_2 t} \tag{3.1}$$

where π_j is the steady state distribution for state *j* given in Lemma 3-1.1 (note that as $t \to \infty$, $P_{ij}(t) \to \pi_j$) and a_{ij} and b_{ij} are constants which we now determine.

Substituting $t = 0$ into equation (3.1) produces,

$$P_{ij}(0) = \delta_{ij} = \pi_j + a_{ij} + b_{ij}$$

or

$$\delta_{ij} - \pi_j = a_{ij} + b_{ij}$$

where $\delta_{ij} = \begin{Bmatrix} 0 & \text{for} & i \neq j \\ 1 & \text{for} & i = j \end{Bmatrix}$ is the Kronecker delta function

For any specific $i, j = 0, 1, 2$,

$$q_{ij} = P'_{ij}(t) \Big|_{t=0} = a_{ij} r_1 e^{r_1 t}\Big|_{t=0} + b_{ij} r_2 e^{r_2 t}\Big|_{t=0}$$

or

$$q_{ij} = a_{ij} r_1 + b_{ij} r_2$$

Then $q_{ij} = a_{ij} r_1 + b_{ij} r_2$ and $\delta_{ij} - \pi_j = a_{ij} + b_{ij}$ can be used to determine a_{ij} and b_{ij}:

$$\delta_{ij} - \pi_j = \frac{q_{ij} - b_{ij} r_2}{r_1} + b_{ij} \quad \rightarrow \quad b_{ij} = \frac{r_1(\delta_{ij} - \pi_j) - q_{ij}}{(r_1 - r_2)}$$

$$q_{ij} = a_{ij} r_1 + \left[\frac{r_1(\delta_{ij} - \pi_j) - q_{ij}}{(r_1 - r_2)}\right] r_2 \quad \rightarrow \quad a_{ij} = \frac{q_{ij} - r_2(\delta_{ij} - \pi_j)}{(r_1 - r_2)}$$

In summary, $P_{ij}(t) = \pi_j + a_{ij} e^{r_1 t} + b_{ij} e^{r_2 t}$ becomes, for Case 1, $B^2 - 4C > 0$

$$P_{ij}(t) = \pi_j + \left[\frac{q_{ij} - r_2(\delta_{ij} - \pi_j)}{(r_1 - r_2)}\right] e^{r_1 t} + \left[\frac{r_1(\delta_{ij} - \pi_j) - q_{ij}}{(r_1 - r_2)}\right] e^{r_2 t}$$

with

$$\pi_0 = \frac{\mu_1\mu_2 + \gamma\lambda_1 + \gamma\mu_1}{C}$$

$$\pi_1 = \frac{\lambda_0\mu_2 + \gamma\lambda_0 + \beta\mu_2}{C}$$

$$\pi_2 = \frac{\lambda_0\lambda_1 + \beta\lambda_1 + \beta\mu_1}{C}$$

$$B = \beta + \gamma + \lambda_0 + \lambda_1 + \mu_1 + \mu_2$$

$$C = \beta(\lambda_1 + \mu_1 + \mu_2) + \gamma(\lambda_0 + \lambda_1 + \mu_1) + \lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_2\mu_1$$

$$r_1,\ r_2 = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$

where the δ_{ij} and q_{ij} are all known.

Case 2 $B^2 - 4C = 0$, double root

In this case,

$$r = \frac{-B \pm \sqrt{B^2 - 4C}}{2} = \frac{-B \pm \sqrt{0}}{2} = \frac{-B}{2}$$

are double roots of the characteristic equation, where $r < 0$ and

$$B = \beta + \gamma + \lambda_0 + \lambda_1 + \mu_1 + \mu_2 .$$

The transient probability functions for double roots are (see [14]),

$$P_{ij}(t) = \pi_j + c_{ij}e^{rt} + d_{ij}te^{rt} \tag{3.2}$$

where π_j is again the steady state distribution for state *j* given in Lemma 3-1.1.

Hence, c_{ij} and d_{ij} are once again constants which may now be determined.

Substituting $t = 0$ into (3.2) gives,

$$\delta_{ij} = P_{ij}(0) = \pi_j + c_{ij}$$

$$c_{ij} = \delta_{ij} - \pi_j$$

where

$$\delta_{ij} = \begin{Bmatrix} 0 & \text{for} & i \neq j \\ 1 & \text{for} & i = j \end{Bmatrix}$$

Differentiating (3.2) and evaluating at t =0 gives

$$q_{ij} = P'_{ij}(t)\ \Big|_{t=0} = c_{ij}(-r)e^{-rt} + d_{ij}\left[e^{-rt} + te^{-rt}(-r)\right]\Big|_{t=0}$$

$$q_{ij} = c_{ij}(-r) + d_{ij}$$

Using $c_{ij} = \delta_{ij} - \pi_j$ we find d_{ij} to be

$$d_{ij} = r(\delta_{ij} - \pi_j) + q_{ij}$$

Thus the transient probability functions, $P_{ij}(t) = \pi_j + c_{ij}e^{rt} + d_{ij}te^{rt}$ for $i, j = 0,1,2$

become, for Case 2 ,

$$\boxed{P_{ij}(t) = \pi_j + \left[\delta_{ij} - \pi_j\right]e^{\frac{-B}{2}t} + \left[-\frac{B}{2}(\pi_j - \delta_{ij}) + q_{ij}\right]t\,e^{\frac{-B}{2}t}}$$

with

$$\pi_0 = \frac{\mu_1\mu_2 + \gamma\lambda_1 + \gamma\mu_1}{C}$$

$$\pi_1 = \frac{\lambda_0\mu_2 + \gamma\lambda_0 + \beta\mu_2}{C}$$

$$\pi_2 = \frac{\lambda_0\lambda_1 + \beta\lambda_1 + \beta\mu_1}{C}$$

$$B = \beta + \gamma + \lambda_0 + \lambda_1 + \mu_1 + \mu_2$$

$$C = \beta(\lambda_1 + \mu_1 + \mu_2) + \gamma(\lambda_0 + \lambda_1 + \mu_1) + \lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_2\mu_1$$

where δ_{ij} and q_{ij} are all known.

Case 3 $B^2 - 4C < 0$

Here,

$$r_1, r_2 = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$

$$= \frac{-B}{2} \pm \frac{i\sqrt{4C - B^2}}{2}$$

where

$$B = \beta + \gamma + \lambda_0 + \lambda_1 + \mu_1 + \mu_2$$

$$C = \beta(\lambda_1 + \mu_1 + \mu_2) + \gamma(\lambda_0 + \lambda_1 + \mu_1) + \lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_2\mu_1$$

Let $r_1 = a + ib$ and $r_2 = a - ib$ where

$$a = \frac{-B}{2}$$

$$b = \frac{\sqrt{4C - B^2}}{2}$$

Hence,

$$P_{jk}(t) = \pi_k + e_{jk} e^{r_1 t} + f_{jk} e^{r_2 t}$$

$$= \pi_k + e_{jk} e^{(a+ib)t} + f_{jk} e^{(a-ib)t}$$

$$= \pi_k + e_{jk} e^{at} e^{ibt} + f_{jk} e^{at} e^{-ibt}$$

Using Euler's relation, $e^{i\theta} = \cos\theta + i\sin\theta$

$$P_{jk}(t) = \pi_k + e_{jk} e^{at}\left(\cos bt + i\sin bt\right) + f_{jk} e^{at}\left(\cos bt - i\sin bt\right)$$

which may be rewritten as

$$P_{jk}(t) = \pi_k + g_{jk} e^{at} \cos bt + h_{jk} e^{at} i \sin bt \qquad (3.3)$$

Setting $t = 0$ in equation (3.3) gives

$$\delta_{jk} = \pi_k + g_{jk} \quad \rightarrow \quad g_{jk} = \delta_{jk} - \pi_k$$

Since π_k is real and δ_{jk} is real, that makes g_{jk} real. Differentiating (3.3) with respect to t and setting $t = 0$,

$$P_{jk}'(t) \Big|_{t=0} = q_{jk} = a \cdot g_{jk} + h_{jk} \cdot b \cdot i$$

But we know q_{jk} is real, g_{jk} is real, b is real, and i is the imaginary unit so that makes h_{jk} pure imaginary, since otherwise we would get a complex number for q_{jk} the transition rate which can't happen. So,

$$-h_{jk} \cdot b \cdot i = a \cdot g_{jk} - q_{jk}$$

$$h_{jk} \cdot b = i \cdot a \cdot g_{jk} - i \cdot q_{jk}$$

$$h_{jk} = \frac{i \cdot a \cdot g_{jk} - i \cdot q_{jk}}{b}$$

$$h_{jk} = i\left[\frac{a \cdot (\delta_{jk} - \pi_k) - q_{jk}}{b}\right]$$

Thus,

$$P_{jk}(t) = \pi_k + g_{jk}e^{at}\cos bt + h_{jk}e^{at} i \sin bt$$

$$= \pi_k + (\delta_{jk} - A_k)e^{at}\cos bt + \left[i\left[\frac{a \cdot (\delta_{jk} - \pi_k) - q_{jk}}{b}\right]\right]e^{at} i \sin bt$$

$$= \pi_k + (\delta_{jk} - \pi_k)e^{at}\cos bt - \left[\frac{a \cdot (\delta_{jk} - \pi_k) - q_{jk}}{b}\right]e^{at}\sin bt$$

$$= \pi_k + (\delta_{jk} - \pi_k)e^{at}\cos bt + \left[\frac{q_{jk} - a \cdot (\delta_{jk} - \pi_k)}{b}\right]e^{at}\sin bt$$

$$= \pi_k + (\delta_{jk} - \pi_k)e^{-\frac{B}{2}t}\cos\left(\frac{\sqrt{4C - B^2}}{2}\right)t + \left[\frac{2q_{jk}}{\sqrt{4C - B^2}} + \frac{B(\delta_{jk} - \pi_k)}{\sqrt{4C - B^2}}\right]e^{-\frac{B}{2}t}\sin\left(\frac{\sqrt{4C - B^2}}{2}t\right)$$

Thus the transient probability function, for Case 3, $B^2 - 4C < 0$ is

$$P_{jk}(t) = \pi_k + \left[\delta_{jk} - \pi_k\right] e^{-\frac{B}{2}t} \cos\left(\frac{\sqrt{4C-B^2}}{2}t\right) + \left[\frac{2q_{jk}}{\sqrt{4C-B^2}} + \frac{B\left(\delta_{jk}-\pi_k\right)}{\sqrt{4C-B^2}}\right] e^{-\frac{B}{2}t} \sin\left(\frac{\sqrt{4C-B^2}}{2}t\right)$$

with

$$\pi_0 = \frac{\mu_1\mu_2 + \gamma\lambda_1 + \gamma\mu_1}{C}$$

$$\pi_1 = \frac{\lambda_0\mu_2 + \gamma\lambda_0 + \beta\mu_2}{C}$$

$$\pi_2 = \frac{\lambda_0\lambda_1 + \beta\lambda_1 + \beta\mu_1}{C}$$

$$B = \beta + \gamma + \lambda_0 + \lambda_1 + \mu_1 + \mu_2$$

$$C = \beta(\lambda_1 + \mu_1 + \mu_2) + \gamma(\lambda_0 + \lambda_1 + \mu_1) + \lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_2\mu_1$$

where δ_{ij} and q_{ij} are all known.

In summary, the transient probability functions for the general three state Markov process with transition rate matrix

$$Q = \begin{bmatrix} -(\lambda_0 + \beta) & \lambda_0 & \beta \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 \\ \gamma & \mu_2 & -(\gamma + \mu_2) \end{bmatrix}$$

are given by

Case 1 For $B^2 - 4C > 0$

$$P_{ij}(t) = \pi_j + \left[\frac{q_{ij} - r_2(\delta_{ij} - \pi_j)}{(r_1 - r_2)}\right] e^{r_1 t} + \left[\frac{r_1(\delta_{ij} - \pi_j) - q_{ij}}{(r_1 - r_2)}\right] e^{r_2 t}$$

Case 2 For $B^2 - 4C = 0$

$$P_{ij}(t) = \pi_j + \left[\delta_{ij} - \pi_j\right] e^{\frac{-B}{2}t} + \left[-\frac{B}{2}(\pi_j - \delta_{ij}) + q_{ij}\right] t\, e^{\frac{-B}{2}t}$$

Case 3 For $B^2 - 4C < 0$

$$P_{jk}(t) = \pi_k + \left[\delta_{jk} - \pi_k\right] e^{-\frac{B}{2}t} \cos\left(\frac{\sqrt{4C-B^2}}{2} t\right) + \left[\frac{2q_{jk}}{\sqrt{4C-B^2}} + \frac{B\left(\delta_{jk} - \pi_k\right)}{\sqrt{4C-B^2}}\right] e^{-\frac{B}{2}t} \sin\left(\frac{\sqrt{4C-B^2}}{2} t\right)$$

with

$$\pi_0 = \frac{\mu_1 \mu_2 + \gamma \lambda_1 + \gamma \mu_1}{C}$$

$$\pi_1 = \frac{\lambda_0 \mu_2 + \gamma \lambda_0 + \beta \mu_2}{C}$$

$$\pi_2 = \frac{\lambda_0 \lambda_1 + \beta \lambda_1 + \beta \mu_1}{C}$$

$$B = \beta + \gamma + \lambda_0 + \lambda_1 + \mu_1 + \mu_2$$

$$C = \beta(\lambda_1 + \mu_1 + \mu_2) + \gamma(\lambda_0 + \lambda_1 + \mu_1) + \lambda_0 \lambda_1 + \lambda_0 \mu_2 + \mu_2 \mu_1$$

$$r_1,\ r_2 = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$

where δ_{ij} and q_{ij} are all known.

Examples

Example 1

Case 1 $B^2 - 4C > 0$

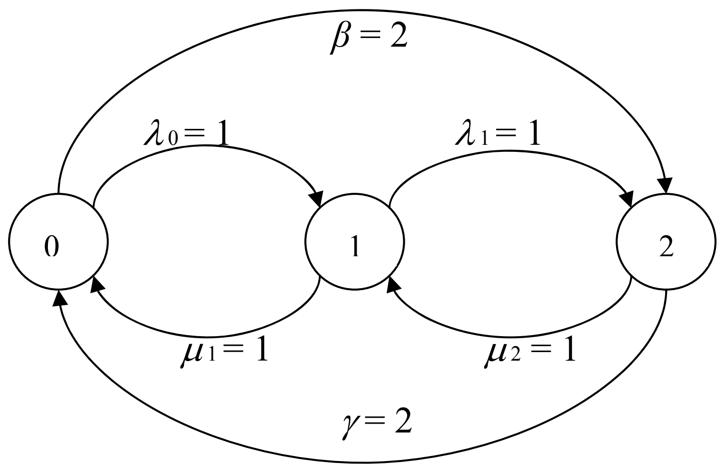

Figure 14

Here,

$$Q = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

and

$$B^2 - 4C = 8^2 - 4[2\cdot 3 + 2\cdot 3 + 3]$$
$$64 - 60 = 4$$

So the roots are,

$$r_1, r_2 = \frac{-8 \pm 2}{2} = -3, -5$$

The steady state distribution is

$$\pi_0 = \pi_1 = \pi_2 = \frac{1}{3}$$

And

$$P_{ij}(t) = \pi_j + \left[\frac{q_{ij} - r_2(\delta_{ij} - \pi_j)}{(r_1 - r_2)}\right]e^{r_1 t} + \left[\frac{r_1(\delta_{ij} - \pi_j) - q_{ij}}{(r_1 - r_2)}\right]e^{r_2 t}$$

$$= \pi_j + \left[\frac{q_{ij} + 5(\delta_{ij} - \pi_j)}{2}\right]e^{-3t} + \left[\frac{-3(\delta_{ij} - \pi_j) - q_{ij}}{2}\right]e^{-5t}$$

For $i = 0$ and $j = 0$,

$\delta_{00} = 1$ and $q_{00} = -3$

$$P_{00}(t) = \frac{1}{3} + \frac{1}{6}e^{-3t} + \frac{1}{2}e^{-5t}$$

For $i = 0$ and $j = 1$,

$\delta_{01} = 0$ and $q_{01} = 1$

$$P_{01}(t) = \frac{1}{3} - \frac{1}{3}e^{-3t}$$

For $i = 0$ and $j = 2$,

$\delta_{02} = 0$ and $q_{02} = 0$

$$P_{02}(t) = \frac{1}{3} - \frac{5}{6}e^{-3t} + \frac{1}{2}e^{-5t}$$

Example 2

Case 2 $B^2 - 4C = 0$

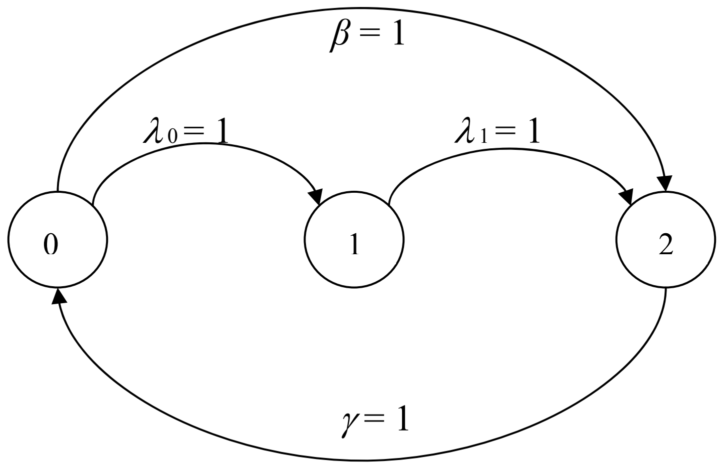

Figure 15

Here

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

The roots of the characteristic equation are

$$r_1, r_2 = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4}}{2}$$
$$= -2$$

So, -2 is a double.

The steady state distribution is

$$\pi_0 = \frac{1}{4},\ \pi_1 = \frac{1}{4},\ \text{and}\ \pi_2 = \frac{1}{2}$$

with $C = 4$ and $B = 4$

and

$$P_{ij}(t) = \pi_j + \left[\delta_{ij} - \pi_j\right] e^{\frac{-B}{2}t} + \left[-\frac{B}{2}(\pi_j - \delta_{ij}) + q_{ij}\right] t\, e^{\frac{-B}{2}t}$$

$$= \pi_j + \left[\delta_{ij} - \pi_j\right] e^{-2t} + \left[-2(\pi_j - \delta_{ij}) + q_{ij}\right] t\, e^{-2t}$$

For $i = 0$ and $j = 0$,

$\delta_{00} = 1$ and $q_{00} = -2$

$$P_{00}(t) = \frac{1}{4} + \frac{3}{4}e^{-2t} - \frac{1}{2}te^{-2t}$$

For $i = 0$ and $j = 1$,

$\delta_{01} = 0$ and $q_{01} = 1$

$$P_{01}(t) = \frac{1}{4} - \frac{1}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$

For $i = 0$ and $j = 2$,

$\delta_{02} = 0$ and $q_{02} = 1$

$$P_{02}(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} + 2te^{-2t}$$

Example 3

Case 3 $B^2 - 4C < 0$

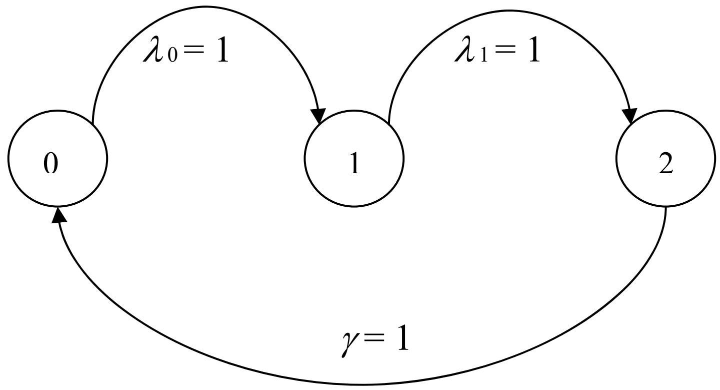

Figure 16

Here,

$$Q = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

The roots of the characteristic equation are

$$r_1, r_2 = \frac{-3 \pm \sqrt{3} \cdot i}{2}$$
$$= \frac{-3}{2} \pm \frac{\sqrt{3}}{2} \cdot i$$

The steady state distribution is

$$\pi_0 = \pi_1 = \pi_2 = \frac{1}{3}$$

with $C = 3$ and $B = 3$

$$P_{jk}(t) = \pi_k + \left[\delta_{jk} - \pi_k\right] e^{-\frac{B}{2}t} \cos\left(\frac{\sqrt{4C-B^2}}{2} t\right) + \left[\frac{2q_{jk}}{\sqrt{4C-B^2}} + \frac{B\left(\delta_{jk} - \pi_k\right)}{\sqrt{4C-B^2}}\right] e^{-\frac{B}{2}t} \sin\left(\frac{\sqrt{4C-B^2}}{2} t\right)$$

$$= \frac{1}{3} + \left[\delta_{jk} - \frac{1}{3}\right] e^{-\frac{3}{2}t} \cos\left(\frac{\sqrt{3}}{2} t\right) + \left[\frac{2q_{jk}}{\sqrt{3}} + \frac{3\left(\delta_{jk} - \frac{1}{3}\right)}{\sqrt{3}}\right] e^{-\frac{3}{2}t} \sin\left(\frac{\sqrt{3}}{2} t\right)$$

For $j = 0$ and $k = 0$,

$\delta_{00} = 1$ and $q_{00} = -1$

$$P_{00}(t) = \frac{1}{3} + \left(\frac{2}{3}\right) e^{-\frac{3}{2}t} \cos\left(\frac{\sqrt{3}}{2} t\right)$$

For $j = 0$ and $k = 1$,

$\delta_{01} = 0$ and $q_{01} = 1$

$$P_{01}(t) = \frac{1}{3} - \frac{1}{3} e^{-\frac{3}{2}t} \cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{1}{\sqrt{3}} e^{-\frac{3}{2}t} \sin\left(\frac{\sqrt{3}}{2} t\right)$$

For $j = 0$ and $k = 2$,

$\delta_{02} = 0$ and $q_{02} = 0$

$$P_{02}(t) = \frac{1}{3} - \frac{1}{3} e^{-\frac{3}{2}t} \cos\left(\frac{\sqrt{3}}{2} t\right) - \frac{1}{\sqrt{3}} e^{-\frac{3}{2}t} \sin\left(\frac{\sqrt{3}}{2} t\right)$$

Conclusion

In this thesis, ruin probabilities are determined for the gambler's ruin with catastrophes and windfalls. For the finite time problem, lattice path combinatorics described in Chapter 1 play a key role in determining the ruin probabilities by providing us a formula for counting sample paths of the Markov chain. In the infinite time case (Chapter 2), the ruin probability recurrence relations have been solved using probability generating functions and the theory of difference equations. These problems generalize and include the well-known, classical gambler's ruin problem as a special case of the solutions developed within this thesis. A preliminary literature search indicates that the solutions determined in this thesis appear new. However, the gambler's ruin problem dates back so many years (over three centuries) that a more extensive literature search still needs to be undertaken. At the same time, further generalizations of the gambler's ruin problems have been progressing at Cal Poly Pomona, see, for example [5] where ruin probabilities having state dependent probabilities are being studied. The gambler's ruin may also be extended along the lines of a batch queueing system, that is, having a transition diagram where a player moves down by two steps instead of one step.

In Chapter 3, the transient probability functions of the general three state Markov process is determined. Our method of solution involves obtaining general, explicit formulae for the eigenvalues of Q and the steady state distribution of the three state Markov process. After this, a categorization of the (distinct) solution function forms is given in Chapter 3.

Even though this approach to determining transient probability functions is well known, so far, we have not seen the solution of the three state Markov process carried out in such complete generality. Our general solution raises some interesting questions that are not yet completely understood. For example, why does the quantity

$$C = \beta(\lambda_1 + \mu_1 + \mu_2) + \gamma(\lambda_0 + \lambda_1 + \mu_1) + \lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_2\mu_1$$

appear in both the steady state distribution of the system and the eigenvalues of Q ? What is the significance of C? The hope is that our general form of solution reveals new insights into the patterns of how transient probability functions look in general. The next step of future research might be to perform a similar analysis for the four state Markov process. The eigenvalues of Q once again may be determined explicitly in terms of the entries of Q -this time by using the formula for the roots of a cubic equation. The steady state distribution is again solvable but not obvious. Since the four state Markov process contains the three and two state Markov process as special cases, again one may hope to learn more about how the general N-state Markov process transient probability looks. In short, the range of future work on this topic is endless.

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