1 Problem 1.6.7

The vectors $u_1 = (2,-3,1)$, $u_2 = (1,4,-2)$, $u_3 = (-8,12,-4)$, $u_4 = (1,37,-17)$, and $u_5 = (-3,-5,8)$ generate $\mathbb{R}^3$. Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

**Solution:** To do so, it suffices to find a linearly independent subset. This is easily done by picking $u_1$ and $u_2$, which are clearly independent, and verifying independence with the other vectors. Doing so gives that $u_5$ is independent from $u_1$ and $u_2$:

$$
\begin{bmatrix}
2 & 1 & -3 \\
-3 & 4 & -5 \\
1 & -2 & 8
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1/2 & -3/2 \\
0 & 1 & -1/11 \\
1 & -2 & 8
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1/2 & 3/2 \\
0 & 1 & -1/11 \\
0 & -5/2 & 13/2
\end{bmatrix}
$$

which reduces, of course, to,

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

which is sufficient to show independence. Thus, the subset $\{u_1, u_2, u_5\}$ is a basis for $\mathbb{R}^3$.

2 Problem 1.6.13

The set of solutions to the system,

\begin{align*}
  x_1 - 2x_2 + x_3 &= 0 \\
  2x_1 - 3x_2 + x_3 &= 0
\end{align*}

**Solution:** Well, first, we notice that if we add the second equation’s negation to the first, we have,

$$
-x_1 + x_2 = 0
$$

in other words, the system is satisfied for $x_1, x_2$ so that $x_1 = x_2$. $x_3$ is seen to depend on the choice of $x_1$, since plugging in $x_1$ for $x_2$ gives that the system is satisfied for $x_1 = x_3$. What emerges is that the solution set of this system is,

$$
S = \text{span}(1,1,1)
$$

which is, in fact, a subspace of $\mathbb{R}^3$. 

3 Problem 1.6.19

Complete the proof of Theorem 1.8.

**Proof:** It remains to be seen that (using the same notation as in the text), if each \( v \in V \) can be uniquely represented as a linear combination of vectors of \( \beta \), then \( \beta \) is a basis of \( V \). Suppose the vectors in \( \beta \) are not linearly independent. Then there exist scalars \( \alpha_i \) not all zero such that,

\[
\sum_{i=1}^{n} \alpha_i \beta_i = 0
\]

But notice that \( \alpha_i = 0 \) for all \( i \) also solves the equation. Thus, 0 has at least two different linear representations, which contradicts uniqueness. Thus, \( \beta \) is linearly independent. Now, we need to show that \( V = \text{span}(\beta) \). If \( v \in V \), then \( v \) can be represented as a linear combination of elements in \( \beta \) and thus \( v \in \text{span}(\beta) \). If \( v \in \text{span}(\beta) \), then obviously \( v \in V \) by closure of \( V \) under addition and scalar multiplication. Thus, \( v = \text{span}(\beta) \) and \( \beta \) is a basis for \( V \).

4 Problem 2.1.2

Let \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) be defined by \( T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) \). Prove that \( T \) is linear and find bases for \( N(T) \) and \( R(T) \). Then, compute the nullity and rank of \( T \), and verify the dimension theorem. Finally, use the appropriate theorems to determine whether \( T \) is injective or surjective.

**Solution:** Let’s start with linearity. We take two vectors, \( (a, b, c) \) and \( (x, y, z) \) in \( \mathbb{R}^3 \). Then \( T(a + x, b + y, c + z) = (a + x - b - y, 2c + 2z) \) by definition. Then, take \( T(a, b, c) + T(x, y, z) \). This is just \( (a - b, 2c) + (x - y, 2z) = (a + x - b - y, 2c + 2z) \). Finally, take \( T(ka, kb, kc) \) for \( k \in F \). This is \( (ka - kb, 2kc) \). Also, \( kT(a, b, c) = k(a - b, 2c) = (ka - kb, 2kc) \). We conclude that \( T \) is linear by definition.

Now, for \( N(T) \), suppose \( T(a, b, c) = 0 \). Then \( (a - b, 2c) = 0 \). Thus, for any vector \( (a, b, c) \) such that \( a = b \) and \( c = 0 \), we have that \( (a, b, c) \in N(T) \). Hence, the basis is \( N(T) = \text{span}(1, 1, 0) \).

Now, we claim that any vector in \( \mathbb{R}^2 \) can be written using transformed elements of \( \mathbb{R}^3 \). Well, take \( (x, y) \in \mathbb{R}^2 \). Then we want to see if,

\[
(x, y) = (a - b, 2c)
\]

for some real \( a, b, c \). But the equation \( a - b = x \) has infinitely many solutions. Finally, if \( \frac{1}{2}y = c \), then we have the desired result. Since the range is all of \( \mathbb{R}^2 \), we can simply use the standard basis as a basis for \( R(T) \).

Finally, we notice that \( T \) is not injective since it’s nullspace does not consist only of the zero vector. However, by the previous argument, the transformation is onto (since its range is all of \( \mathbb{R}^2 \)).

5 Problem 2.1.15

Recall the definition of \( P(\mathbb{R}) \) given on page 10 of the text. Define \( T : P(\mathbb{R}) \to P(\mathbb{R}) \) by,
\[ T(f(x)) = \int_0^x f(t)\,dt \]

Prove that \( T \) is linear and injective, but not surjective.

**Proof:** Linearity is straightforward. Take \( f, g \in P(\mathbb{R}) \). Then,

\[ T(f + g) = \int_0^x (f + g)(t)\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt = T(f) + T(g) \]

In the most pure sense, however, this is not completely justified. To justify this completely, you need at least one academic quarter of real analysis and a thorough understanding of *partitions* and *integrability* (it suffices, in this case, to just understand the Riemann Integral; yes, there are other types of integrals!). For now, just rely on the properties you learned in elementary calculus.

Now, we show injectivity. Suppose \( T(f) = T(g) \) for some \( f, g \). Then,

\[ \int_0^x f(t)\,dt = \int_0^x g(t)\,dt \]

Now, we have the following,

\[ \int_0^x f(t)\,dt - \int_0^x g(t)\,dt = 0 \]

By the fundamental theorem of calculus (which you can assume), we differentiate both sides

\[ f(x) - g(x) = 0 \]

Thus, since \( f(x) = g(x) \), we have proven injectivity. Another way of doing this is by showing that the nullspace is zero. This is equally valid:

\[ \int_0^x f(t)\,dt = 0 \]

differentiating,

\[ f(x) = 0 \]

for arbitrary \( f(x) \). Thus, \( N(T) = \{0\} \); that is, the zero function. However, it is not onto. To see this, notice that for any constant function \( c \), there exists no function in \( P(\mathbb{R}) \) so that \( P(f) = c \).
6 Problem 2.1.18

Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

Solution: An example of such is this. Let,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

That is, for any $(x, y)$, $T(x, y) = (y, 0)$. Then we can see that $N(T) = \text{span}(1, 0)$ and so is $R(T)$.

7 Online Problem 1

Let $T : V \to W$ be a linear map, show the null space, null$(T)$, is a subspace of $V$.

Proof: Choose $x, y \in (T)$. Choose $c \in F$. Notice that $0 = T(x) + T(y) = T(x + y)$. Hence, $x + y \in \text{null}(T)$. But also, $0 = cT(x) = T(cx)$. Thus, $cx \in \text{null}(T)$. We conclude that null$(T)$ is a subspace by definition.

8 Problem 2.2.4

Define $T : M_{2 \times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ by,

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + b) + (2d)x + bx^2$$

Let,

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and $\gamma = \{1, x, x^2\}$. Compute $[T]_\beta^\gamma$.

Solution: Apply $T$ to the basis elements to get $1, 1 + x^2, 0$, and $2x$ respectively. Then, if $\beta_i$ represent the various elements of $\beta$, we take,

$$[T]_\beta^\gamma = \left( [T(\beta_1)]_\gamma, [T(\beta_2)]_\gamma, [T(\beta_3)]_\gamma, [T(\beta_4)]_\gamma \right)$$

This is,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
9 Problem 2.2.5a

Let,
\[
\alpha = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
and \(\beta = \{1, x, x^2\}\). Finally, let \(\gamma = \{1\}\). Define \(T : M_{2 \times 2}(F) \to M_{2 \times 2}(F)\) by \(T(A) = A^t\). Compute \([T]_{\alpha}\).

**Solution:** This is straightforward. All we must do, again, is apply the transformation to the basis elements. Let \(\alpha_i\) denote the basis elements of \(\alpha\). Then \(T(\alpha_1)\) and \(T(\alpha_4)\) remain unchanged. However, \(T(\alpha_2) = \alpha_3\). Also, \(T(\alpha_3) = T(\alpha_2)\). The matrix, then, is,
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

10 Problem 2.2.16

Let \(V\) and \(W\) be vector spaces such that \(\dim(V) = \dim(W)\), and \(T : V \to W\) be linear. Show that there exist ordered bases \(\beta\) and \(\gamma\) for \(V\) and \(W\) respectively, such that \([T]_{\gamma}^{\beta}\) is a diagonal matrix.

**Proof:** Let \(\alpha = \{v_1, ..., v_k\}\) be an ordered basis for \(\text{N}(T)\). Extend \(\alpha\) to an ordered basis for the whole space: \(\beta = \{v_1, ..., v_k, v_{k+1}, ..., v_n\}\). Now, we write the vectors \(\{T(v_{k+1}), ..., T(v_n)\}\) as are an ordered basis for \(\text{R}(T)\) (as per the proof of the dimension theorem). Extend this to the ordered basis \(\gamma = \{w_1, ..., w_k, T(v_{k+1}), ..., T(v_n)\}\) for \(W\). Now,
\[
[T(v_i)]_\gamma = \begin{cases}
0 & \text{if } 1 \leq i \leq k \\
e_i & \text{if } k + 1 \leq i \leq n
\end{cases}
\]
This gives the desired result,
\[
[T]_\gamma^{\beta} = (0, ..., 0, e_{k+1}, ..., e_k)
\]
That is,
\[
\begin{bmatrix}
0 & 0 \\
0 & I_{\frac{n-k}{2} \times \frac{n-k}{2}}
\end{bmatrix}
\]