MATH 30 - 2 & 3, SPRING 2018, PRACTICE TEST I

Disclaimer: This practice test DOES NOT serve as an indication of the contents of the actual test. It only suggests a possible format.

Please print your name clearly!

Name: SOLUTIONS

Please show all your work, that is explain every step of your solution - it is your work, not the answer, that is being evaluated. When asked to prove a statement, make sure to provide reasoning behind each claim you are making in the process of the proof. The use of calculators or any other electronic devices is prohibited during the test. You are also not allowed to use any study materials except for those provided to you during the test. Cheating is strictly prohibited, and will be prosecuted. Good luck!

Problem 1 (20 points). Are the following functions even, odd, or neither? Explain.

a) - (5 points) $f(x) = x \sin x$

b) - (5 points) $g(x) = \frac{(x+1)(x-1)}{(e^x+e^{-x})\cos x}$

c) - (5 points) $h(x) = (x + |x|)\tan x$

d) - (5 points) $t(x) = \frac{x}{|x|}$

Solution. a) Notice that, since $\sin(-x) = -\sin x$,

$$f(-x) = (-x)\sin(-x) = (-x)(-\sin x) = x \sin x = f(x),$$

hence $f(x)$ is an even function.

b) Notice that, since $\cos(-x) = \cos x$,

$$\begin{aligned} g(-x) &= \frac{((-x)+1)((-x)-1)}{(e^{-x}+e^{-(-x)})\cos(-x)} = \frac{-(-x+1)(x+1)}{(e^{-x}+e^{x})\cos x} \\ &= \frac{(x-1)(x+1)}{(e^{-x}+e^{x})\cos x} = g(x), \end{aligned}$$

hence $g(x)$ is an even function.

c) Notice that, since $\tan(-x) = -\tan x$,

$h(-x) = (-x + |-x|)\tan(-x) = -(-x + |x|)\tan x = (x - |x|)\tan x$,

i.e. $h(-x)$ is not equal to $h(x)$ or to $-h(x)$. Therefore $h(x)$ is not an even or an odd function.

d) Notice that

$$t(-x) = \frac{-x}{|-x|} = -\frac{x}{|x|} = -t(x),$$

hence $t(x)$ is an odd function.

Problem 2 (20 points). Find inverses of the following functions, if they exist. For each inverse, specify the domain and range.

a) - (10 points) $f(x) = \sqrt{x} - 7$

b) - (10 points) $g(x) = x^4 - 5$, $x \leq 0$

Solution. a) Let $y = \sqrt{x} - 7$, then $\sqrt{x} = y + 7$, and so

$$x = (y+7)^2 = y^2 + 14y + 49.$$

Switching x and y, we obtain:

$$f^{-1}(x) = x^2 + 14x + 49.$$

The domain of $f^{-1}(x)$ is the range of $f(x)$, which is $[-7, \infty)$. The range of $f^{-1}(x)$ is the domain of $f(x)$, which is $[0, \infty)$.

b) Let $y = x^4 - 5$, then $x^4 = y + 5$, and since $x \leq 0$,

$$x = -\sqrt[4]{y+5}.$$

Switching x and y, we obtain:

$$g^{-1}(x) = -\sqrt[4]{x+5}.$$

The domain of $g^{-1}(x)$ is the range of $g(x)$, which is $[-5, \infty)$. The range of $g^{-1}(x)$ is the domain of $g(x)$, which is $(-\infty, 0]$.

Problem 3 (20 points). Find the specified values of inverse trigonometric functions.

a) - (10 points) $\tan^{-1}\left(\tan\left(\frac{27\pi}{3}\right)\right)$

b) - (10 points) $\sin^{-1}\left(\sin(\pi)+\cos\left(\frac{\pi}{3}\right)\right)$

Solution. a) Let $\theta = \tan^{-1}\left(\tan\left(\frac{27\pi}{3}\right)\right)$, then

$$\tan\theta = \tan\left(\frac{27\pi}{3}\right) = \tan(9\pi) = \tan\pi = 0,$$

and $-\pi/2 \leq \theta \leq \pi/2$. Hence $\theta = 0$, i.e.

$$\tan^{-1}\left(\tan\left(\frac{27\pi}{3}\right)\right) = 0.$$

b) Notice that

$$\sin\pi + \cos\frac{\pi}{3} = 0 + \frac{1}{2} = \frac{1}{2}.$$

Letting $\theta = \sin^{-1}\left(\sin(\pi)+\cos\left(\frac{\pi}{3}\right)\right)$, we see that $\sin\theta = \frac{1}{2}$ and $-\pi/2 \leq \theta \leq \pi/2$. Hence $\theta = \pi/6$, i.e.

$$\sin^{-1}\left(\sin(\pi)+\cos\left(\frac{\pi}{3}\right)\right) = \pi/6.$$

Problem 4 (20 points). Compute the following limits or show that they do not exist.

a) - (10 points) $\lim_{x\to 4} \frac{x^2-2x-8}{x^2-x-12}$

b) - (10 points) $\lim_{x\to -3} \frac{x+3}{|x+3|}$

Solution. a)

$$\begin{aligned}\lim_{x\to 4} \frac{x^2-2x-8}{x^2-x-12} &= \lim_{x\to 4} \frac{(x-4)(x+2)}{(x-4)(x+3)} \\ &= \lim_{x\to 4} \frac{x+2}{x+3} = \frac{4+2}{4+3} = \frac{6}{7}.\end{aligned}$$

Hence the answer is 6/7.

b) First suppose that $x < -3$, then $|x+3| = -(x+3)$, and so

$$\lim_{x\to -3^-} \frac{x+3}{|x+3|} = \lim_{x\to -3^-} \frac{x+3}{-(x+3)} = -1.$$

Next assume that $x > -3$, then $|x+3| = x+3$, and so

$$\lim_{x\to -3^+} \frac{x+3}{|x+3|} = \lim_{x\to -3^-} \frac{x+3}{x+3} = 1.$$

Therefore

$$\lim_{x\to -3^-} \frac{x+3}{|x+3|} \neq \lim_{x\to -3^+} \frac{x+3}{|x+3|},$$

and hence $\lim_{x\to -3} \frac{x+3}{|x+3|}$ does not exist.

Problem 5 (20 points). Find the domains of the following functions.

a) - (10 points) $f(x) = e^{-\sec x}$

b) - (10 points) $g(x) = \ln(\sin x)$

Solution. a) Recall that $\sec x = \frac{1}{\cos x}$, and so domain of $\sec x$ consists of all the real numbers at which $\cos x \neq 0$, which is the set

$$\{x \in \mathbb{R} : x \neq \pi/2 + \pi n, n \in \mathbb{Z}\}.$$

Since e^x is defined everywhere, this is the domain of $f(x)$.

b) Recall that $\sin x$ is defined everywhere, however $\ln x$ is only defined when the argument is positive. Now, $\sin x$ is positive on all intervals of the form

$$(0, \pi) + 2\pi n, n \in \mathbb{Z}.$$

Hence the domain of $g(x)$ is the union of all such intervals:

$$\bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi) = \ldots (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \ldots$$

Problem 6 (20 points). Use the epsilon-delta definition of the limit to prove that

$$\lim_{x\to -1} \frac{x^2 - x - 2}{x+1} = -3.$$

Solution. First notice that

$$\frac{x^2 - x - 2}{x+1} = \frac{(x-2)(x+1)}{x+1} = x - 2,$$

since $x + 1 \neq 0$ (in our limit, x tends to -1, but it is not equal to -1). Hence we need to prove that

$$\lim_{x\to -1} (x-2) = -3.$$

Let $\varepsilon > 0$, then we want

$$|(x-2) - (-3)| < \varepsilon.$$

Observe that $(x-2) - (-3) = x - (-1)$, and so we can take $\delta = \varepsilon$. Then whenever

$$|x - (-1)| < \delta,$$

we have

$$|(x-2) - (-3)| < \varepsilon,$$

and so $\lim_{x\to -1}(x-2) = -3$.