MATH 195, SPRING 2015, TAKE-HOME MIDTERM

Please print your name clearly!

Name:____

This test is due on Tuesday, 3/31/15, in class. While completing it, feel free to use your lecture notes from class, as well as those posted on the class webpage. You are however not allowed to consult with anyone: it is understood that solutions to this midterm represent solely your own work with no outside assistance. Good luck!

Problem 1. (10 points) Let n be a positive integer. A complex number

$$\zeta_n = e^{\frac{2\pi i}{n}}$$

is called *n-th primitive root of unity*. Notice that $\zeta_n^n = e^{2\pi i} = 1$ by Euler's formula, and hence ζ_n is algebraic: it is a root of the polynomial $x^n - 1$.

Let $n \geq 2$, and consider the following argument. First notice that

$$\ln \zeta_n = \frac{2\pi i}{n}.$$

On the other hand,

$$0 = \ln 1 = \ln \zeta_n^n = n \ln \zeta_n,$$

and hence $\ln \zeta_n = 0$. But obviously $\frac{2\pi i}{n} \neq 0$. What went wrong? Explain your answer.

Problem 2. (10 points) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find an integral primitive element for K, i.e. an element $\alpha \in \mathcal{O}_K$ such that $K = \mathbb{Q}(\alpha)$. Prove your answer.

Problem 3. (20 points) A subset S of $\mathbb{R}$ is called *discrete* if there exists real $\varepsilon > 0$ such that for every two distinct elements $\alpha, \beta \in S$,

$$|\alpha - \beta| \geq \varepsilon.$$

On the other hand, let us say that S is *near-discrete* if for every $\alpha \in S$ there exist a real $\varepsilon = \varepsilon(\alpha) > 0$ such that

$$|\alpha - \beta| \geq \varepsilon$$

for every $\beta \in S$ distinct from α.

Part 1. (10 points) Prove that every discrete subset of $\mathbb{R}$ is countable.

Part 1. (10 points) Prove that every near-discrete subset of $\mathbb{R}$ is countable.

Problem 4. (20 points) Let $\mu > 0$. We say that $p/q \in \mathbb{Q}$ is a μ-approximation to the real number α if

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^{\mu}}.$$

Let $S \subseteq \mathbb{R}$ be a set with the following properties:

(1) Every element of S has infinitely many rational 3-approximations.
(2) If a rational number p/q is a 3-approximation for some $\alpha \in S$, then it is not a 3-approximation for any other element of S.

Part 1. (10 points) Prove that S is countable.

Part 2. (10 points) Prove that every $\alpha \in S$ is transcendental.

Problem 5. (20 points) Let K be a subfield of $\mathbb{C}$. An embedding of K into $\mathbb{C}$ is an injective field homomorphism $\tau : K \to \mathbb{C}$.

Part 1. (10 points) Suppose that $\tau : K \to \mathbb{C}$ is a field homomorphism such that for some $a \in K$, $\tau(a) \neq 0$. Prove that τ is an embedding.

Part 2. (10 points) Let $K = \mathbb{Q}(\alpha)$ for some $\alpha \in \mathbb{C}$. Prove that α is transcendental if and only if there exist infinitely many distinct embeddings of K into $\mathbb{C}$.

Problem 6. (30 points) Let R be a subring of the ring S, both commutative rings with identity. Let $\alpha \in S$, and define

$$R[\alpha] = \{f(\alpha) : f(x) \in R[x]\}.$$

More generally, for $\alpha_1, \dots, \alpha_n \in S$ define recursively

$$R[\alpha_1, \dots, \alpha_n] = R_{n-1}[\alpha_n],$$

where $R_{n-1} = R[\alpha_1, \dots, \alpha_{n-1}]$. A subring T of S is called a finitely generated ring extension of R if $T = R[\alpha_1, \dots, \alpha_n]$ for some $\alpha_1, \dots, \alpha_n \in S$.

Part 1. (5 points) Prove that $R[\alpha]$ is a subring of S.

Part 2. (5 points) Prove that $R[\alpha]$ is the smallest ring containing R and α, with respect to inclusion.

Part 3. (10 points) Let $T = \mathbb{Z}[1/n]$ for some integer $n > 1$. Compute the group of units $T^\times$ of T.

Part 4. (10 points) Prove that $\mathbb{Q}$ is not a finitely generated ring extension of $\mathbb{Z}$.

Problem 7. (20 points) Let $\{n_j\}_{j=1}^{\infty}$ be a sequence of natural numbers such that

$$\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = \infty.$$

Define

$$f(x) = \sum_{j=1}^{\infty} x^{n_j},$$

and let α be a real algebraic number such that $0 < \alpha < 1$. Prove that $f(\alpha)$ is transcendental.