Uniformity of the Möbius function in $\mathbf{F}_q[t]$

Thái Hoàng Lê[1] & Pierre-Yves Bienvenu[2]

[1]University of Mississippi

[2]University of Bristol

January 13, 2018

The Möbius randomness principle

Recall

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } n \text{ is not squarefree.} \end{cases}$$

Thus the sequence $\{\mu(n)\}$ is $1, -1, -1, 0, -1, 1, \ldots$.

The Möbius randomness principle

Recall

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } n \text{ is not squarefree.} \end{cases}$$

Thus the sequence $\{\mu(n)\}$ is $1, -1, -1, 0, -1, 1, \ldots$.

The **Möbius randomness principle** states that μ is random-like,

The Möbius randomness principle

Recall

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } n \text{ is not squarefree.} \end{cases}$$

Thus the sequence $\{\mu(n)\}$ is $1, -1, -1, 0, -1, 1, \ldots$.

The **Möbius randomness principle** states that μ is random-like, i.e. for any bounded, “simple” or “structured” function F, we have

$$\sum_{n=1}^{N} \mu(n) F(n) = o(N).$$

Examples:

1. If $F(n) = 1$, then PNT is equivalent to $\sum_{n=1}^{N} \mu(n) = o(N)$ and *RH* is equivalent to

$$\sum_{n=1}^{N} \mu(n) = O_\epsilon\left(N^{1/2+\epsilon}\right)$$

for any $\epsilon > 0$.

Examples:

1. If $F(n) = 1$, then PNT is equivalent to $\sum_{n=1}^{N} \mu(n) = o(N)$ and *RH* is equivalent to
$$\sum_{n=1}^{N} \mu(n) = O_\epsilon\left(N^{1/2+\epsilon}\right)$$
for any $\epsilon > 0$.

2. If $F(n)$ is periodic with period q, then $\sum_{n=1}^{N} \mu(n)F(n) = o(N)$ is equivalent to PNT in arithmetic progressions.

3. Sarnak's conjecture: If (X, T) is a dynamical system (i.e. X is a compact metric space, $T : X \to X$ is continuous) with topological entropy zero, then for any $x \in X$ and $f \in C(X)$,

$$\sum_{n=1}^{N} \mu(n) f(T^n x) = o(N).$$

3. Sarnak's conjecture: If (X, T) is a dynamical system (i.e. X is a compact metric space, $T : X \to X$ is continuous) with topological entropy zero, then for any $x \in X$ and $f \in C(X)$,

$$\sum_{n=1}^{N} \mu(n) f(T^n x) = o(N).$$

4. (Kalai/Green 2011) If F can be computed by bounded depth circuits, then

$$\sum_{n=1}^{N} \mu(n) F(n) = o(N).$$

3. Sarnak's conjecture: If (X, T) is a dynamical system (i.e. X is a compact metric space, $T : X \to X$ is continuous) with topological entropy zero, then for any $x \in X$ and $f \in C(X)$,

$$\sum_{n=1}^{N} \mu(n) f(T^n x) = o(N).$$

4. (Kalai/Green 2011) If F can be computed by bounded depth circuits, then

$$\sum_{n=1}^{N} \mu(n) F(n) = o(N).$$

5. (Dartyge-Tenenbaum 2005) If $s(n)$ is the sum of digits of n in base q, then for any $\alpha \in \mathbf{R}/\mathbf{Z}$,

$$\sum_{n=1}^{N} \mu(n) e(s(n)\alpha) = o(N).$$

Here $e(x) = e^{2\pi i x}$.

Exponential sums

Davenport (1937): for any $A > 0$,

$$\sum_{n=1}^{N} \mu(n) e(n\alpha) \ll_A \frac{N}{\log^A N}$$

uniformly in $\alpha \in \mathbf{R}/\mathbf{Z}$. The implied constant is ineffective.

Davenport (1937): for any $A > 0$,

$$\sum_{n=1}^{N} \mu(n) e(n\alpha) \ll_A \frac{N}{\log^A N}$$

uniformly in $\alpha \in \mathbf{R}/\mathbf{Z}$. The implied constant is ineffective.

Baker-Harman (1991), Montgomery-Vaughan (unpublished): Assuming GRH, we have

$$\sum_{n=1}^{N} \mu(n) e(n\alpha) \ll_\epsilon N^{3/4+\epsilon}$$

uniformly in $\alpha \in \mathbf{R}/\mathbf{Z}$, for any $\epsilon > 0$.

Since

$$\int_0^1 \left| \sum_{n=1}^{N} \mu(n) e(n\alpha) \right|^2 = \sum_{n=1}^{N} |\mu(n)|^2 \gg N,$$

we cannot do better than $N^{1/2}$.

Since

$$\int_0^1 \left| \sum_{n=1}^{N} \mu(n) e(n\alpha) \right|^2 = \sum_{n=1}^{N} |\mu(n)|^2 \gg N,$$

we cannot do better than $N^{1/2}$.

Green-Tao (2008, 2012): If $F(n)$ is a nilsequence, then for any $A > 0$,

$$\sum_{n=1}^{N} \mu(n) F(n) \ll_A \frac{N}{\log^A N}.$$

Since

$$\int_0^1 \left|\sum_{n=1}^{N} \mu(n)e(n\alpha)\right|^2 = \sum_{n=1}^{N} |\mu(n)|^2 \gg N,$$

we cannot do better than $N^{1/2}$.

Green-Tao (2008, 2012): If $F(n)$ is a nilsequence, then for any $A > 0$,

$$\sum_{n=1}^{N} \mu(n)F(n) \ll_A \frac{N}{\log^A N}.$$

Nilsequences include $F(n) = e\left(\alpha n^2 + \beta n\right)$ and $F(n) = e\left(\lfloor n\alpha \rfloor \beta n\right)$.

Function field analogy

Let $\mathbf{F}_q$ be the finite field on q elements. $\mathbf{F}_q[t]$ is similar to $\mathbf{Z}$ in many aspects (both are UFDs).

Function field analogy

Let $\mathbf{F}_q$ be the finite field on q elements. $\mathbf{F}_q[t]$ is similar to $\mathbf{Z}$ in many aspects (both are UFDs).

For $f \in \mathbf{F}_q[t]$, define

$$\mu(f) = \begin{cases} (-1)^k & \text{if } f = cP_1P_2\cdots P_k,\ P_i \text{ distinct monic irreducibles,} \\ & \quad c \in \mathbf{F}_q^\times, \\ 0 & \text{if } f \text{ is not squarefree.} \end{cases}$$

Function field analogy

Let $\mathbf{F}_q$ be the finite field on q elements. $\mathbf{F}_q[t]$ is similar to $\mathbf{Z}$ in many aspects (both are UFDs).

For $f \in \mathbf{F}_q[t]$, define

$$\mu(f) = \begin{cases} (-1)^k & \text{if } f = cP_1P_2\cdots P_k,\ P_i \text{ distinct monic irreducibles,} \\ & \qquad c \in \mathbf{F}_q^{\times}, \\ 0 & \text{if } f \text{ is not squarefree.} \end{cases}$$

RH is true in $\mathbf{F}_q[t]$: for $n \geq 2$,

$$\sum_{\substack{\deg f = n, \\ f \text{ monic}}} \mu(f) = 0.$$

Function field analogy

Let $\mathbf{F}_q$ be the finite field on q elements. $\mathbf{F}_q[t]$ is similar to $\mathbf{Z}$ in many aspects (both are UFDs).

For $f \in \mathbf{F}_q[t]$, define

$$\mu(f) = \begin{cases} (-1)^k & \text{if } f = cP_1P_2\cdots P_k,\ P_i \text{ distinct monic irreducibles,} \\ & \qquad c \in \mathbf{F}_q^{\times}, \\ 0 & \text{if } f \text{ is not squarefree.} \end{cases}$$

RH is true in $\mathbf{F}_q[t]$: for $n \geq 2$,

$$\sum_{\substack{\deg f = n, \\ f \text{ monic}}} \mu(f) = 0.$$

Furthermore, GRH is true in $\mathbf{F}_q[t]$ (Weil 1948).

The field of fractions $\mathbf{F}_q(t) = \{f/g : f, g \in \mathbf{F}_q[t], g \neq 0\}$ is the analog of $\mathbf{Q}$.

The field of fractions $\mathbf{F}_q(t) = \{f/g : f, g \in \mathbf{F}_q[t], g \neq 0\}$ is the analog of $\mathbf{Q}$.

Define $|f/g| = q^{\deg f - \deg g}$. The completion of $\mathbf{F}_q(t)$ with respect to $|\cdot|$ is

$$\mathbf{F}_q\left(\left(\frac{1}{t}\right)\right) = \left\{\alpha = \sum_{i=-\infty}^{n} a_i t^i : n \in \mathbf{Z}, a_i \in \mathbf{F}_q \text{ for every } i\right\},$$

the set of formal Laurent series in $\frac{1}{t}$. This is the analog of $\mathbf{R}$.

The field of fractions $\mathbf{F}_q(t) = \{f/g : f, g \in \mathbf{F}_q[t], g \neq 0\}$ is the analog of $\mathbf{Q}$.

Define $|f/g| = q^{\deg f - \deg g}$. The completion of $\mathbf{F}_q(t)$ with respect to $|\cdot|$ is

$$\mathbf{F}_q\left(\left(\frac{1}{t}\right)\right) = \left\{\alpha = \sum_{i=-\infty}^{n} a_i t^i : n \in \mathbf{Z}, a_i \in \mathbf{F}_q \text{ for every } i\right\},$$

the set of formal Laurent series in $\frac{1}{t}$. This is the analog of $\mathbf{R}$.

Define $\mathbf{T} = \{\alpha \in \mathbf{F}_q\left(\left(\frac{1}{t}\right)\right) : |\alpha| < 1\}$. This is the analog of $\mathbf{R}/\mathbf{Z}$.

Fix $e_q : \mathbf{F}_q \to \{z \in \mathbf{C} : |z| = 1\}$ to be an additive character of $\mathbf{F}_q$.

Fix $e_q : \mathbf{F}_q \to \{z \in \mathbf{C} : |z| = 1\}$ to be an additive character of $\mathbf{F}_q$.

For $\alpha = \sum_{i=-\infty}^{n} a_i t^i \in \mathbf{F}_q((\frac{1}{t}))$, define

$$e(\alpha) = e_q(a_{-1}).$$

Fix $e_q : \mathbf{F}_q \to \{z \in \mathbf{C} : |z| = 1\}$ to be an additive character of $\mathbf{F}_q$.

For $\alpha = \sum_{i=-\infty}^{n} a_i t^i \in \mathbf{F}_q((\frac{1}{t}))$, define

$$e(\alpha) = e_q(a_{-1}).$$

Thus $e : \mathbf{F}_q((\frac{1}{t})) \to \{z \in \mathbf{C} : |z| = 1\}$ is an additive character of $\mathbf{F}_q((\frac{1}{t}))$.

Problems we want to solve

Problem 1. Let $k \geq 1$ and $\Phi(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \cdots + \alpha_1 x$ be a polynomial in $\mathbf{F}_q((\frac{1}{t}))[x]$. Show that

$$\sum_{\deg f < n} \mu(f) e(\Phi(f)) = o(q^n)$$

uniformly in Φ of degree k. Better yet, (since we have GRH) obtain a bound $O(q^{c_k n})$ for some constant $c_k < 1$.

Problems we want to solve

Problem 1. Let $k \geq 1$ and $\Phi(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \cdots + \alpha_1 x$ be a polynomial in $\mathbf{F}_q((\frac{1}{t}))[x]$. Show that

$$\sum_{\deg f < n} \mu(f) e(\Phi(f)) = o(q^n)$$

uniformly in Φ of degree k. Better yet, (since we have GRH) obtain a bound $O(q^{c_k n})$ for some constant $c_k < 1$.

There are differences between this and the integer case. In fact, when $k \geq p = \mathrm{char}(\mathbf{F}_q)$, even the Weyl sum $\sum_{\deg f < n} e(\Phi(f))$ is not well understood, since the Weyl differencing process breaks down ($k! = 0$).

Problem 2. Let $k \geq 1$ and $Q \in \mathbf{F}_q[x_0, x_1, \ldots, x_{n-1}]$ be a polynomial of degree k. Show that

$$\sum_{\deg f < n} \mu(f) e_q(Q(f)) = o(q^n)$$

uniformly in Q of degree k. Here $Q(f)$ is Q evaluated at the coefficients of f. Better yet, (since we have GRH) obtain a bound $O(q^{c_k n})$ for some constant $c_k < 1$.

Problem 2. Let $k \geq 1$ and $Q \in \mathbf{F}_q[x_0, x_1, \ldots, x_{n-1}]$ be a polynomial of degree k. Show that

$$\sum_{\deg f < n} \mu(f) e_q(Q(f)) = o(q^n)$$

uniformly in Q of degree k. Here $Q(f)$ is Q evaluated at the coefficients of f. Better yet, (since we have GRH) obtain a bound $O(q^{c_k n})$ for some constant $c_k < 1$.

Problem 2 is more general than Problem 1 since for any polynomial $\Phi(x)$, $(\Phi(f))_{-1}$ is a polynomial in the coefficients of f (recall that $e(\alpha) = e_q((\alpha)_{-1})$).

The linear case

When $k = 1$ then Problem 1 and Problem 2 are the same. This is because for any linear form $\ell \in \mathbf{F}_q[x_0, \ldots, x_{n-1}]$, there is $\alpha \in \mathbf{T}$ such that $\ell(f) = (\alpha f)_{-1}$ for any $\deg f < n$.

The linear case

When $k = 1$ then Problem 1 and Problem 2 are the same. This is because for any linear form $\ell \in \mathbf{F}_q[x_0, \dots, x_{n-1}]$, there is $\alpha \in \mathbf{T}$ such that $\ell(f) = (\alpha f)_{-1}$ for any $\deg f < n$.

Theorem

For any $\epsilon > 0$, we have

$$\sum_{\deg f < n} \mu(f) e(\alpha f) \ll_{\epsilon, q} q^{(3/4+\epsilon)n}$$

uniformly in $\alpha \in \mathbf{T}$.

Recall Baker-Harman and Montgomery-Vaughan's bound in **Z** (under GRH)

$$\sum_{n=1}^{N} \mu(n) e(\alpha n) \ll N^{3/4+\epsilon}.$$

Recall Baker-Harman and Montgomery-Vaughan's bound in **Z** (under GRH)

$$\sum_{n=1}^{N} \mu(n)e(\alpha n) \ll N^{3/4+\epsilon}.$$

Porritt independently proved a similar result which gives the same exponent $3/4 + \epsilon$ when q is sufficiently large in terms of ϵ.

Recall Baker-Harman and Montgomery-Vaughan's bound in $\mathbf{Z}$ (under GRH)

$$\sum_{n=1}^{N} \mu(n) e(\alpha n) \ll N^{3/4+\epsilon}.$$

Porritt independently proved a similar result which gives the same exponent $3/4+\epsilon$ when q is sufficiently large in terms of ϵ.

Our argument is different from the proof in $\mathbf{Z}$ in some respects. This is because the topologies of $\mathbf{T}$ and $\mathbf{R}/\mathbf{Z}$ are different and there is no summation by parts in $\mathbf{F}_q[t]$.

We use Hayes' L-functions of *arithmetically distributed relations*, as opposed to Dirichlet L-functions. Let M be the set of monic polynomials in $\mathbf{F}_q[t]$. Define an equivalence relation $f \equiv g \pmod{R_{l,Q}}$ on M as follows

$f \equiv g \pmod{Q}$ and the first $l + 1$ coefficients of f and g are the same.

We use Hayes' L-functions of *arithmetically distributed relations*, as opposed to Dirichlet L-functions. Let M be the set of monic polynomials in $\mathbf{F}_q[t]$. Define an equivalence relation $f \equiv g \pmod{R_{l,Q}}$ on M as follows

$f \equiv g \pmod{Q}$ and the first $l+1$ coefficients of f and g are the same.

Then $M/R_{l,Q}$ is a semigroup, $G_{l,Q} := (M/R_{l,Q})^\times$ is a group. If λ is a nontrivial character of $G_{l,Q}$, then

$$L(s, \lambda) = \sum_{f \in M} \frac{\lambda(f)}{|f|^s}$$

satisfies GRH (Rhin 1972). (If $l = 0$ then λ is a Dirichlet character.)

Theorem

Assuming a quantitative form of the bilinear Bogolyubov theorem and $p > 2$. *Then for any* $A > 0$ *and quadratic polynomial* Q *in* $\mathbf{F}_q[x_0, \dots, x_{n-1}]$,

$$\sum_{\deg f < n} \mu(f) e(Q(f)) \ll_{q,A} q^n n^{-A}$$

uniformly in Q.

The quadratic case

Theorem

There is a constant $c < 1$ such that

$$\sum_{\deg f < n} \mu(f) e(\alpha f^2 + \beta f) \ll_q q^{cn}$$

uniformly in $\alpha, \beta \in \mathbf{T}$.

Ideas of proof

The classical circle method deals with sums like $\sum_{\deg f<n} \mu(f)e(\alpha f)$ or $\sum_{\deg f<n} \mu(f)e(\alpha f^2 + \beta f)$. To estimate such sums we need to distinguish two cases.

- α is close to a rational with small denominator (α is in the *major arcs*): use our understanding of the distribution of irreducibles in congruence classes, i.e. L-functions.

Ideas of proof

The classical circle method deals with sums like $\sum_{\deg f<n} \mu(f)e(\alpha f)$ or $\sum_{\deg f<n} \mu(f)e(\alpha f^2 + \beta f)$. To estimate such sums we need to distinguish two cases.

- α is close to a rational with small denominator (α is in the *major arcs*): use our understanding of the distribution of irreducibles in congruence classes, i.e. *L*-functions.
- α is not in the major arcs (α is in the *minor arcs*): use Vaughan's identity to decompose the sum into Vinogradov's Type I/Type II sums.

In the case of $\sum_{\deg f<n}\mu(f)e_q(\Phi(f))$, where $\Phi(x)=x^TMx$ and M is a symmetric matrix, the major arcs and minor arcs correspond to low rank and high rank matrices M. This is because

$$\left|\sum_{x\in\mathbf{F}_q{}^n} e_q(x^TMx)\right| \leq q^{n-\mathrm{rank}(M)/2}.$$

In the case of $\sum_{\deg f<n} \mu(f) e_q(\Phi(f))$, where $\Phi(x) = x^T Mx$ and M is a symmetric matrix, the major arcs and minor arcs correspond to low rank and high rank matrices M. This is because

$$\left| \sum_{x \in \mathbf{F}_q{}^n} e_q(x^T Mx) \right| \leq q^{n-\mathrm{rank}(M)/2}.$$

The low rank can be easily reduced to the linear case.

In the case of $\sum_{\deg f<n} \mu(f) e_q(\Phi(f))$, where $\Phi(x) = x^T M x$ and M is a symmetric matrix, the major arcs and minor arcs correspond to low rank and high rank matrices M. This is because

$$\left| \sum_{x \in \mathbf{F}_q{}^n} e_q(x^T M x) \right| \leq q^{n - \mathrm{rank}(M)/2}.$$

The low rank can be easily reduced to the linear case.

Suppose for a contradiction that $\sum_{\deg f<n} \mu(f) e_q(\Phi(f)) \geq \delta q^n$. We want to show that $\mathrm{rank}(\mathrm{M})$ is small.

After using Vaughan's identity, Vinogradov's Type I/Type II sums, Cauchy-Schwarz and some combinatorial reasoning, we find that for some $n \ll k \leq n$, the set of pairs

$$P_h := \{(a, b) : \deg a, \deg b \in G_{k+1} \times G_{k+1} : \mathrm{rank}\, M_{a,b} \leq h\}$$

is large (has size $(\delta/n)^{O(1)} q^{2k+2}$) for some $h = O(\log(n/\delta))$.

After using Vaughan's identity, Vinogradov's Type I/Type II sums, Cauchy-Schwarz and some combinatorial reasoning, we find that for some $n \ll k \leq n$, the set of pairs

$$P_h := \{(a, b) : \deg a, \deg b \in G_{k+1} \times G_{k+1} : \mathrm{rank}\, M_{a,b} \leq h\}$$

is large (has size $(\delta/n)^{O(1)} q^{2k+2}$) for some $h = O(\log(n/\delta))$.

Here $G_m = \{f : \deg f < m\}$,

$$M_{a,b} = L_a^T M L_b + L_b^T M L_a$$

and L_a is the matrix of the map $G_{n-k} \to G_n, f \mapsto af$.

Problem: We know

$$P_h := \{(a, b) : \deg a, \deg b \in G_{k+1} \times G_{k+1} : \mathrm{rank}\, M_{a,b} \leq h\}$$

is large, where $M_{a,b} = L_a^T M L_b + L_b^T M L_a$. Show that $\mathrm{rank}\, M$ is small.

Problem: We know

$$P_h := \{(a, b) : \deg a, \deg b \in G_{k+1} \times G_{k+1} : \mathrm{rank}\, M_{a,b} \leq h\}$$

is large, where $M_{a,b} = L_a^T M L_b + L_b^T M L_a$. Show that $\mathrm{rank}\, M$ is small.

If $(a, b), (a', b) \in P_h$, then $(a - a', b) \in P_{2h}$ since $M_{a-a',b} = M_{a,b} - M_{a',b}$. Similarly for the second coordinate.

Problem: We know

$$P_h := \{(a,b) : \deg a, \deg b \in G_{k+1} \times G_{k+1} : \mathrm{rank}\, M_{a,b} \leq h\}$$

is large, where $M_{a,b} = L_a^T M L_b + L_b^T M L_a$. Show that $\mathrm{rank}\, M$ is small.

If $(a,b),(a',b) \in P_h$, then $(a-a',b) \in P_{2h}$ since $M_{a-a',b} = M_{a,b} - M_{a',b}$. Similarly for the second coordinate.

By repeatedly "smoothing" in each coordinate, we find that P_{64h} contains an interesting structure.

Theorem (Bilinear Bogolyubov Theorem)

If $|P_h| \geq \eta q^{2(k+1)}$*, then there exist* $\mathbf{F}_p$*-subspaces* W_1, W_2 *of the* $\mathbf{F}_p$*-vector space* G_{k+1} *of codimension* r_1, r_2 *and* $\mathbf{F}_p$*-bilinear forms* $Q_1, \ldots, Q_r$ *on* $W_1 \times W_2$ *such that*

$$P_{64h} \supset \{(x, y) \in W_1 \times W_2 \mid Q_1(x, y) = \cdots = Q_r(x, y) = 0\}$$

and $\max(r, r_1, r_2) \leq c(\eta)$*.*

Theorem (Bilinear Bogolyubov Theorem)

If $|P_h| \geq \eta q^{2(k+1)}$*, then there exist* $\mathbf{F}_p$*-subspaces* W_1, W_2 *of the* $\mathbf{F}_p$*-vector space* G_{k+1} *of codimension* r_1, r_2 *and* $\mathbf{F}_p$*-bilinear forms* $Q_1, \ldots, Q_r$ *on* $W_1 \times W_2$ *such that*

$$P_{64h} \supset \{(x, y) \in W_1 \times W_2 \mid Q_1(x, y) = \cdots = Q_r(x, y) = 0\}$$

and $\max(r, r_1, r_2) \leq c(\eta)$.

Theorem (Bilinear Bogolyubov Theorem)

If $|P_h| \geq \eta q^{2(k+1)}$*, then there exist* $\mathbf{F}_p$*-subspaces* W_1, W_2 *of the* $\mathbf{F}_p$*-vector space* G_{k+1} *of codimension* r_1, r_2 *and* $\mathbf{F}_p$*-bilinear forms* $Q_1, \ldots, Q_r$ *on* $W_1 \times W_2$ *such that*

$$P_{64h} \supset \{(x,y) \in W_1 \times W_2 \mid Q_1(x,y) = \cdots = Q_r(x,y) = 0\}$$

and $\max(r, r_1, r_2) \leq c(\eta)$*.*

A similar result was proved independently by Gowers-Milicevic.

Theorem (Bilinear Bogolyubov Theorem)

If $|P_h| \geq \eta q^{2(k+1)}$, then there exist $\mathbf{F}_p$-subspaces W_1, W_2 of the $\mathbf{F}_p$-vector space G_{k+1} of codimension r_1, r_2 and $\mathbf{F}_p$-bilinear forms $Q_1, \ldots, Q_r$ on $W_1 \times W_2$ such that

$$P_{64h} \supset \{(x, y) \in W_1 \times W_2 \mid Q_1(x, y) = \cdots = Q_r(x, y) = 0\}$$

and $\max(r, r_1, r_2) \leq c(\eta)$.

A similar result was proved independently by Gowers-Milicevic.

We can take $c(\eta) = O(\exp(\exp(\exp(\log^{O(1)} \eta^{-1}))))$. This is too large to be useful. If we can take $c(\eta) = \log^{O(1)} \eta^{-1}$ then in our application we can take $\delta = n^{-A}$ and this implies that M has small rank.