Higher dimensional Steinhaus and Slater problems via homogeneous dynamics

Alan Haynes
University of Houston

Joint work with Jens Marklof (Bristol)

Steinhaus problem

- For $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$, let

$$S(N) = \{n\alpha \bmod 1 : 1 \leq n \leq N\},$$

  and let $G(N)$ denote the number of distinct lengths of component intervals of $(\mathbb{R}/\mathbb{Z}) \setminus S(N)$.

- Theorem (Sós 1957, Surányi 1958, Świerczkowski 1959): For any $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$, we have that $G(N) \leq 3$.

Higher dimensional version

- For $\alpha, \beta \in \mathbb{R}$ and $M, N \in \mathbb{N}$ let

$$S(M,N) = \{m\alpha + n\beta \bmod 1 : 1 \leq m \leq M, 1 \leq n \leq N\},$$

  and let $G(M,N)$ denote the number of distinct lengths of component intervals of $(\mathbb{R}/\mathbb{Z}) \setminus S(M,N)$.

- Question: Given α and β, what can we say about the size of $G(M,N)$?

- Previously studied by: Erdős; Geelen and Simpson; Fraenkel and Holzman; Chevallier; Bleher, Homma, Ji, Roeder, and Shen; Boshernitzan and Dyson.

First bounds for $G(M, N)$

- Theorem (Geelen, Simpson, 1993): For any α, β, M, N,

$$G(M, N) \leq \min(M, N) + 3.$$

- Proposition: If $A, B \in \mathbb{Z}$ and $A\alpha + B\beta \in \mathbb{Z}$ then for any M, N,

$$G(M, N) \leq \min(|A|, |B|) + 2|AB| + 3.$$

- Problem (Erdős, Geelen, Simpson): If $1, \alpha$, and β are $\mathbb{Q}$-linearly independent, does it follow that

$$\sup_{N \in \mathbb{N}} G(N, N) = \infty\ ?$$

- Theorem (BD; BHJRS, 2011):

$$\dim_H \{(\alpha, \beta) \in \mathbb{R}^2 : \sup_{N \in \mathbb{N}} G(N, N) < \infty\} = 2.$$

Unbounded gaps and the Littlewood conjecture

- Problem: Do there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\sup_{M,N\in\mathbb{N}} G(M,N) = \infty \text{ ?}$$

- Proposition (H., Marklof, 2017): If the above equation holds for some α and β then it follows that

$$\inf_{n\in\mathbb{N}} n\|n\alpha\|\|n\beta\| = 0.$$

- Notation: $\|x\| = \min_{n\in\mathbb{N}} |x-n|$.

Lower bound for gaps in two dimensions

- Theorem (H., Marklof, 2017): For almost every $(\alpha, \beta) \in \mathbb{R}^2$,

$$\sup_{N \in \mathbb{N}} G(N, N) = \infty.$$

- Remark 1: This is stronger than concluding that

$$\sup_{M, N \in \mathbb{N}} G(M, N) = \infty.$$

- Remark 2: Our proof also works with the $N \times N$ square replaced by $N\Omega$, for any convex Ω with non-empty interior. It also generalizes to any dimension ≥ 2.

- Our proof is a development of an argument that was used by Marklof and Strömbergsson to give a geometric proof of the classical Steinhaus problem.

Sketch of proof: Elementary arguments

- For any $1 \le m \le M$ and $1 \le n \le N$, the length of the component interval of $(\mathbb{R}/\mathbb{Z}) \setminus S(M, N)$ having $m\alpha + n\beta$ as its left endpoint is

$$\begin{aligned}
&\min \{(m' - m)\alpha + (n' - n)\beta + \ell > 0 : \\
&\qquad 0 < m' \le M, 0 < n' \le N, \ell \in \mathbb{Z}\} \\
&= \min \left\{ m'\alpha + n'\beta + \ell > 0 : \right. \\
&\qquad \left. \frac{-m}{M} < \frac{m'}{M} \le 1 - \frac{m}{M}, \frac{-n}{N} < \frac{n'}{N} \le 1 - \frac{n}{N}, \ell \in \mathbb{Z} \right\}.
\end{aligned}$$

Sketch of proof: Elementary arguments

- For each α, β, M, and N, let

$$\mathcal{M} = \mathcal{M}_{\alpha,\beta,M,N} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M^{-1} & 0 & 0 \\ 0 & N^{-1} & 0 \\ 0 & 0 & MN \end{pmatrix}.$$

- The set $\mathbb{Z}^3\mathcal{M}$ is a lattice in $\mathbb{R}^3$ of covolume 1, with lattice points given by

$$\left(\frac{m'}{M}, \ \frac{n'}{N}, \ MN(m'\alpha + n'\beta + \ell)\right),$$

for $m', n', \ell \in \mathbb{Z}$.

Sketch of proof: Elementary arguments

- It follows that for any $1 \leq m \leq M$ and $1 \leq n \leq N$, the length of the component interval of $(\mathbb{R}/\mathbb{Z}) \setminus S(M, N)$ having $m\alpha + n\beta$ as its left endpoint is $(MN)^{-1}$ times

$$\min \Big\{ x_3 > 0 : (x_1, x_2, x_3) \in \mathbb{Z}^3 \mathcal{M}, \\ \frac{-m}{M} < x_1 \leq 1 - \frac{m}{M}, \ \frac{-n}{N} < x_2 \leq 1 - \frac{n}{N} \Big\}.$$

Sketch of proof: Dynamical formulation

- Let
$$X_3 = \mathrm{SL}_3(\mathbb{Z}) \setminus \mathrm{SL}_3(\mathbb{R}),$$
and define $F : X_3 \times [0,1)^2 \to \mathbb{R}$ by
$$F(\mathcal{L}, t) = \min \big\{ x_3 > 0 : (x_1, x_2, x_3) \in \mathbb{Z}^3 \mathcal{L}, \\ -t_1 < x_1 \leq 1 - t_1, \ \ -t_2 < x_2 \leq 1 - t_2 \big\}.$$
- Next, for each $\mathcal{L} \in X_3$, let $G(\mathcal{L})$ be the number of distinct values taken by $F(\mathcal{L}, t)$, as t runs over $[0,1)^2$.
- For any α, β, M, and N we have that
$$G(M, N) = G(\mathrm{SL}_3(\mathbb{Z}) \mathcal{M}_{\alpha,\beta,M,N}).$$

Sketch of proof: Dynamical formulation

- The flow $\phi_s : X_3 \to X_3$ defined by
$$\mathcal{L} \mapsto \mathcal{L} \cdot \begin{pmatrix} e^{-s} & 0 & 0 \\ 0 & e^{-s} & 0 \\ 0 & 0 & e^{2s} \end{pmatrix}, \quad s \in \mathbb{R},$$
is ergodic with respect to Haar measure on X_3.
- There is a set $P \subset \mathbb{R}^2$ of full measure such that for $(\alpha, \beta) \in P$,
$$\overline{\left\{ \phi_s \left(\mathrm{SL}_3(\mathbb{Z}) \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \right) \ : \ s \geq 0 \right\}} = X_3.$$
- This in turn implies that
$$\overline{\left\{ \phi_{\log N} \left(\mathrm{SL}_3(\mathbb{Z}) \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \right) \ : \ N \in \mathbb{N} \right\}} = X_3.$$

Sketch of proof: Unboundedness of $G(\cdot)$

- Let $\epsilon \in (0, 1/2)$ be irrational, and define $\mathcal{L}_\epsilon \in X_3$ by

$$\mathcal{L}_\epsilon = \mathrm{SL}_3(\mathbb{Z}) \cdot \begin{pmatrix} -\epsilon & -\epsilon & \epsilon \\ 1 & 1 & 0 \\ 0 & 1/\epsilon & 0 \end{pmatrix}.$$

- Points in $\mathbb{Z}^3 \mathcal{L}_\epsilon$ have the form

$$(-a_1\epsilon + a_2, -a_1\epsilon + a_2 + a_3/\epsilon, a_1\epsilon),$$

with $a_1, a_2, a_3 \in \mathbb{Z}$.

Sketch of proof: Unboundedness of $G(\cdot)$

- It follows that there is a neighborhood $U_\epsilon \subseteq X_3$ of $\mathcal{L}_\epsilon$ with the property that

$$G(\mathcal{L}) \geq \lfloor 1/\epsilon \rfloor \quad \text{for all} \quad \mathcal{L} \in U_\epsilon.$$

- This together with the observations on the previous slides completes the proof of the main theorem.

Some remarks

- Our proof does not tell us how to explicitly construct α and β for which
$$\sup_{M,N} G(M,N) = \infty.$$
However, there is a recent explicit construction due to Valérie Berthé and Dong Han Kim.
- It would be interesting to know whether or not, with $\alpha = (1+\sqrt{5})/2, \beta = \sqrt{2}$, is it true that
$$\sup_{M,N\in\mathbb{N}} G(M,N) = \infty \ .$$

Slater problem

- For $\alpha \in \mathbb{R}^d$, consider the toral translation

$$\mathbb{T}^d \mapsto \mathbb{T}^d, \qquad q \mapsto q + \alpha,$$

  where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.
- Let $\mathcal{D} \subset \mathbb{R}^d$ be a convex open set which is contained in a bounded fundamental domain of $\mathbb{R}^d/\mathbb{Z}^d$. e.g. $\mathcal{D} \subset (-\frac{1}{2}, \frac{1}{2}]^d$.
- For $q \in \mathcal{D}$, the first return time to $\mathcal{D}$ is given by

$$\tau(q, \mathcal{D}) = \min\{n \in \mathbb{N} \mid q + n\alpha \in \mathcal{D} + \mathbb{Z}^d\}.$$

Slater's Theorem

- $L(\alpha, \mathcal{D})$ be the number of distinct values $\tau(q, \mathcal{D})$ attains, as q varies over $\mathcal{D}$.
- Theorem (Slater, 1950): For $d = 1$, any α, and any interval $\mathcal{D}$, $L(\alpha, \mathcal{D}) \leq 3$.

Higher dimensional results

- Theorem (H., Marklof, 2017) For $d \geq 2$ and for a.e. α,

$$\sup_{N \in \mathbb{N}} L(\alpha, N^{-1}\mathcal{D}) = \infty.$$

and

$$\liminf_{N \in \mathbb{N}} L(\alpha, N^{-1}\mathcal{D}) < \infty.$$

Thank you for your time!