Parametric Geometry of Numbers

WOLFGANG M. SCHMIDT
University of Colorado at Boulder

JMM, San Diego, CA
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1 Minkowski’s theorem

\[ K \subset \mathbb{R}^n \text{ compact, convex, symmetric about } 0, \]
and \( \text{vol}(K) = 2^n \)

\[ \implies \text{exists } z \in \mathbb{Z}^n \setminus \{0\} \ (z \in K) \]

May replace \( \mathbb{Z}^n \) by a lattice \( \Lambda \) of covolume (determinant), i.e., the image of \( \mathbb{Z}^n \) under linear map of determinant 1.

2 Successive minima

2.1 Minkowski

\( \lambda_1, \ldots, \lambda_n \) of \( \Lambda \) and \( K \)
(\( \lambda_i \) least so that \( \lambda_i K \) contains \( i \) linearly independent lattice points)

Then:

\[ \lambda_1 \leq \ldots \leq \lambda_n \]
\[ \frac{1}{n!} \leq \lambda_1 \cdots \lambda_n \leq 1 \]
2.2 Schmidt-Summerer

\( K \) unit cube of points \((\eta_1, \ldots, \eta_n)\) with \(|\eta_i| \leq 1\) \((i = 1, 2, \ldots, n)\)

\( \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n \)
\( \nu_1 + \cdots + \nu_n = 0 \)

\( T^q : \mathbb{R}^n \to \mathbb{R}^n \)
\((\eta_1, \ldots, \eta_n) \mapsto (e^{\nu_1 q} \eta_1, \ldots, e^{\nu_n q} \eta_n)\)

\( K(q) = T^q(K) \), a box of volume \(2^n\)

\( \lambda_i(q) \) of \( \Lambda \) and \( K(q) \), \( i = 1, 2, \ldots, n \)

Then:

\[
\frac{1}{n!} \leq \lambda_1(q) \cdots \lambda_n(q) \leq 1
\]

"Parametric Geometry of Numbers" with parameter \( q \)
(L. Summerer, S.)
Set

\[ L_i(q) := \log \lambda_i(q), \ i = 1, 2, \ldots, n. \]

Then:

\[ L_1(q) \leq \ldots \leq L_n(q) \]

\[ -\log n! \leq L_1(q) + \cdots + L_n(q) \leq 0 \]

Each \( L_i(q) \) is piecewise linear with slopes among

\(-\nu_n, \ldots, -\nu_2, -\nu_1.\)

**Example.** \( n = 3, \ \nu = (2, -1, -1), \) slopes 1, 1, -2
3 Conjectures

\[
\varphi_i(q) := \frac{L_i(q)}{q}, \quad i = 1, 2, \ldots, n
\]

\[
\underline{\varphi}_i := \lim \inf \varphi_i(q)
\]

\[
\overline{\varphi}_i := \lim \sup \varphi_i(q)
\]

Note:

\[-\nu_1 \leq \underline{\varphi}_1 \leq \ldots \leq \underline{\varphi}_n\]

\[\overline{\varphi}_1 \leq \ldots \leq \overline{\varphi}_n \leq -\nu_n\]

\[\underline{\varphi}_i \leq \overline{\varphi}_i\]

**Conjecture 3.1.** The set of all arising 2n-tuples

\((\underline{\varphi}_1, \overline{\varphi}_1, \ldots, \underline{\varphi}_n, \overline{\varphi}_n) \in \mathbb{R}^{2n}\) (for given \(\nu\)) is semialgebraic.

Behaviour of the map \(L = (L_1, \ldots, L_n) : \mathbb{R}_{>0} \to \mathbb{R}^n\) could be better.
**Definition.** A map $\mathbf{P} = (P_1, \ldots, P_n) : \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$ is called a $\nu$-system if the following conditions are satisfied:

(i) $P_1(q) \leq \cdots \leq P_n(q)$.

(ii) $P_1(q) + \cdots + P_n(q) = 0$.

(iii) Each $P_i$ is continuous and piecewise linear with slopes among $-\nu_n, \ldots, -\nu_2, -\nu_1$.

(iv) Extra condition. E.g., the following is not allowed.

![Diagram of P_1, P_2, P_3]

**Conjecture 3.2.** Let $\nu$ be given. If $L_1, \ldots, L_n$ arise from a lattice, then there exists a $\nu$-system $\mathbf{P} = (P_1, \ldots, P_n)$ with

\[ |L_i(q) - P_i(q)| \text{ bounded } (i = 1, 2, \ldots, n). \]

Conversely, given a $\nu$-system $\mathbf{P}$, there exists a lattice $\Lambda$ whose $L_1, \ldots, L_n$ satisfy (*).
Damien Roy: Case $n = m + 1$, $\nu = (m, -1, \ldots, -1)$  
In this case:

$$K(q) \ |\eta_1| \leq e^{mq}, \ |\eta_i| \leq e^{-q} \ (i = 2, 3, \ldots, n)$$
\[\Lambda\] to consist of points
\[
(x, \xi_1x - y_1, \ldots, \xi_m x - y_m), \ (x, y_1, \ldots, y_m) \in \mathbb{Z}^n
\]

Note: (i) Lattice points lie in $K(q)$ if

\[(**) \quad |x| \leq e^{mq} \quad \text{and} \quad |\xi_i x - y_i| \leq e^{-q} \quad (i = 1, 2, \ldots, m).
\]

(ii)

$$\text{vol}(K) = 2^n \quad \text{and} \quad \text{cvol}(\Lambda) = 1$$

\[\implies\] exists a lattice point in $K(q)$ (Dirichlet).

Divide $(**)$ by $x$ to get

$$\left| \frac{\xi_i - y_i}{x} \right| \leq \frac{1}{x^{1 + \frac{1}{m}}}, \quad (i = 1, 2, \ldots, m)$$

(Simultaneous approximation to $\xi_1, \ldots, \xi_m$).
Improve Dirichlet?

\( \alpha(p) \) largest, so there exist \( x, y_1, \ldots, y_m \) such that
\[
|x| \leq e^p, \quad |\xi_i x - y_i| \leq e^{-\alpha(p)p} \quad (i = 1, 2, \ldots, n)
\]
\[
\alpha(p) \geq \frac{1}{m}
\]
\[
\alpha := \lim \inf \alpha(p)
\]
\[
\omega := \lim \sup \alpha(p)
\]

Some other notation \((\alpha, \omega), (\bar{\omega}, \omega), (\alpha, \beta)\)

When \( n = 1 \), then \( \alpha = 1 \).

\( \omega \) irrationality exponent of \( \xi \):

\[
\omega = 1 \text{ for } \xi = e \text{ and for } \xi \text{ algebraic (Roth)}
\]
\[
\omega = ? \text{ for } \xi = \pi \text{ (cf. } \omega < 4)\]
Dual quantities $\alpha^*$ and $\omega^*$ for “approximation to a linear form”

Jarnik, Khintchine (1920’s, 1930’s)
— Relations between $\alpha, \omega, \alpha^*, \omega^*$

\[
(1 + \alpha)(m + \varphi_1) = (1 + \omega)(m + \varphi_1) = n \\
(1 + \alpha^*)(1 - \varphi_n) = (1 + \omega^*)(1 - \varphi_n) = n
\]

Jarnik \quad n = 3, \quad \varphi_1 + \varphi_3 + \varphi_1\varphi_3 = 0

Khintchine \quad (n - 1)\varphi_1 + \varphi_n \leq 0 \quad \text{and} \quad (n - 1)\varphi_n + \varphi_1 \geq 0

Further, Bugeaud, Laurent, Moshchevitin, German, S., S.

**Conjecture 3.3.** Any inequality on $\varphi_1, \varphi_1, \ldots, \varphi_n, \varphi_n$ remains valid if $\varphi_i \mapsto \varphi_{n+1-i}$ and $\varphi_i \mapsto \varphi_{n+1-i}$ and the inequality is reversed.