Parametric Geometry of Numbers

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1 Minkowski's theorem

$$\begin{aligned} & K \subset \mathbb{R}^n \text{ compact, convex, symmetric about } \underline{0}, \\ & \text{and } \mathrm{vol}(K) = 2^n \\ \Longrightarrow \quad & \text{exists } \underline{z} \in \mathbb{Z}^n \setminus \{\underline{0}\} \quad (\underline{z} \in K) \end{aligned}$$

May replace $\mathbb{Z}^n$ by a lattice Λ of covolume (determinant), i.e., the image of $\mathbb{Z}^n$ under linear map of determinant 1.

2 Successive minima

2.1 Minkowski

$\lambda_1, \ldots, \lambda_n$ of Λ and K
(λ_i least so that $\lambda_i K$ contains i linearly independent lattice points)

Then:

$$\begin{aligned} \lambda_1 \leq \ldots \leq \lambda_n \\ \frac{1}{n!} \leq \lambda_1 \cdots \lambda_n \leq 1 \end{aligned}$$

2.2 Schimdt-Summerer

$$
\begin{aligned}
K \quad & \text{unit cube of points } (\eta_1, \ldots, \eta_n) \\
& \text{with } |\eta_i| \leq 1 \quad (i = 1, 2, \ldots, n) \\
\underline{\nu} \quad & = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n \\
& \nu_1 + \cdots + \nu_n = 0 \\
T^q \quad & : \mathbb{R}^n \to \mathbb{R}^n \\
& (\eta_1, \ldots, \eta_n) \mapsto (e^{\nu_1 q}\eta_1, \ldots, e^{\nu_n q}\eta_n) \\
K(q) \quad & = T^q(K), \text{ a box of volume } 2^n
\end{aligned}
$$

$$\lambda_i(q) \quad \text{of } \Lambda \text{ and } K(q), \;\; i = 1, 2, \ldots, n$$

Then:

$$\frac{1}{n!} \;\leq\; \lambda_1(q) \cdots \lambda_n(q) \;\leq\; 1$$

"Parametric Geometry of Numbers" with parameter q
(L. Summerer, S.)

Set

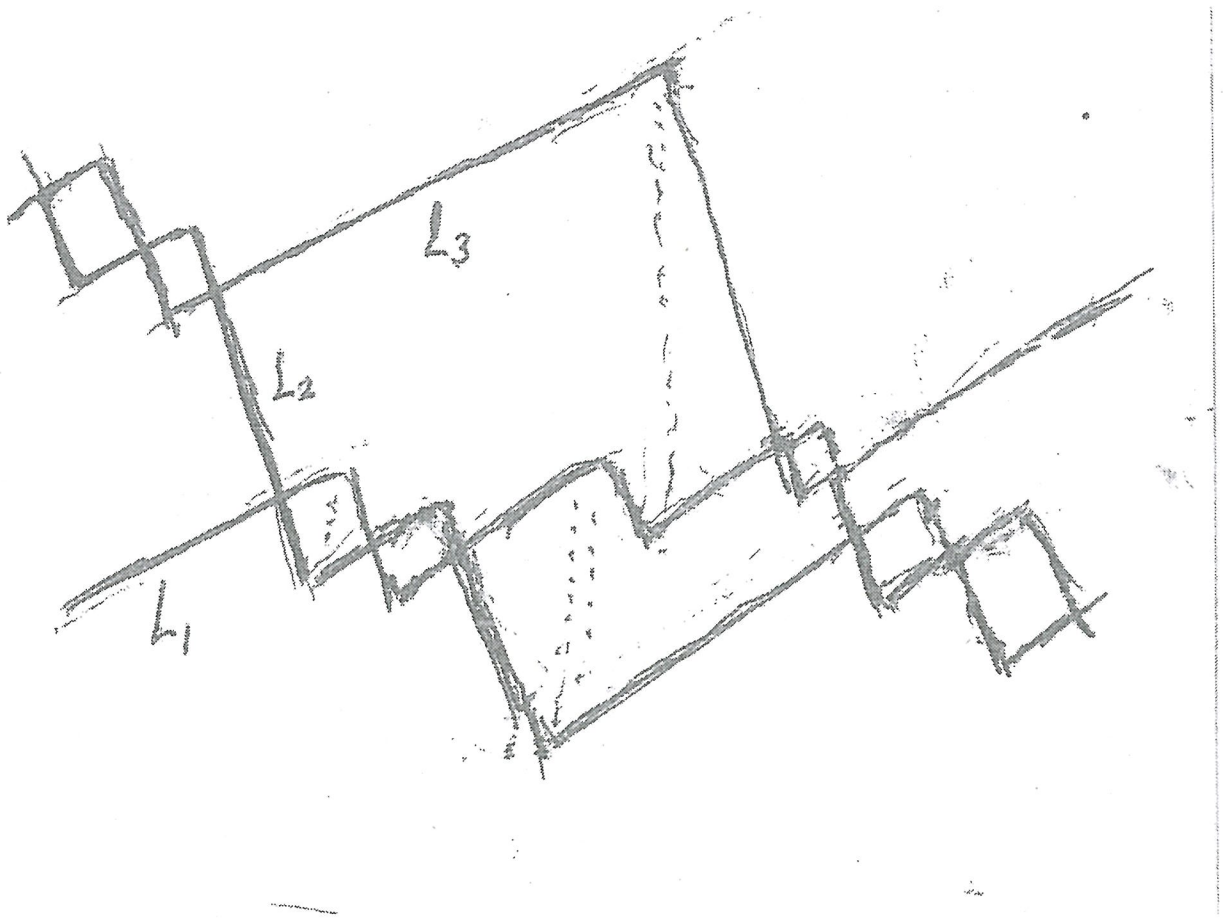
$$L_i(q) \quad := \log \lambda_i(q), \; i = 1, 2, \ldots, n.$$

Then:

$$L_1(q) \;\leq\; \ldots \;\leq\; L_n(q)$$
$$-\log n! \;\leq\; L_1(q) + \cdots + L_n(q) \;\leq\; 0$$

Each $L_i(q)$ is piecewise linear with slopes among $-\nu_n, \ldots, -\nu_2, -\nu_1$.

Example. $n = 3$, $\underline{\nu} = (2, -1, -1)$, slopes $1, 1, -2$



3 Conjectures

$$\begin{aligned}
\varphi_i(q) &:= \frac{L_i(q)}{q}, \ i = 1, 2, \ldots, n \\
\underline{\varphi}_i &:= \liminf \varphi_i(q) \\
\overline{\varphi}_i &:= \limsup \varphi_i(q)
\end{aligned}$$

Note:

$$-\nu_1 \leq \underline{\varphi}_1 \leq \ldots \leq \underline{\varphi}_n$$

$$\overline{\varphi}_1 \leq \ldots \leq \overline{\varphi}_n \leq -\nu_n$$

$$\underline{\varphi}_i \leq \overline{\varphi}_i$$

Conjecture 3.1. *The set of all arising $2n$-tuples $(\underline{\varphi}_1, \overline{\varphi}_1, \ \ldots, \ \underline{\varphi}_n, \overline{\varphi}_n) \in \mathbb{R}^{2n}$ (for given $\underline{\nu}$) is semialgebraic.*

Behaviour of the map $\underline{L} = (L_1, \ldots, L_n) : \mathbb{R}_{>0} \to \mathbb{R}^n$ could be better.

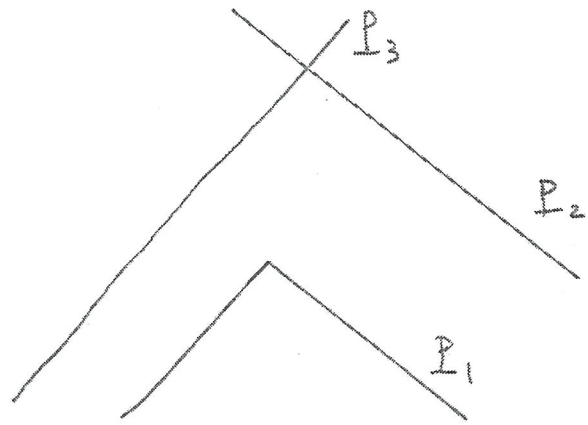
Definition. A map $\underline{P} = (P_1, \ldots, P_n) : \mathbb{R}_{>0} \to \mathbb{R}^n$ is called a *$\underline{\nu}$-system* if the following conditions are satisfied:

(*i*) $P_1(q) \leq \cdots \leq P_n(q)$.

(*ii*) $P_1(q) + \cdots + P_n(q) = 0$.

(*iii*) Each P_i is continuous and piecewise linear with slopes among $-\nu_n, \ldots, -\nu_2, -\nu_1$.

(*iv*) Extra condition. E.g., the following is not allowed.



Conjecture 3.2. *Let $\underline{\nu}$ be given. If $L_1, \ldots, L_n$ arise from a lattice, then there exists a $\underline{\nu}$-system $\underline{P} = (P_1, \ldots, P_n)$ with*

(*) $\quad |L_i(q) - P_i(q)|$ *bounded* $(i = 1, 2, \ldots, n)$.

Conversely, given a $\underline{\nu}$-system $\underline{P}$, there exists a lattice Λ whose $L_1, \ldots, L_n$ satisfy (*).

Damien Roy: Case $n = m+1$, $\underline{\nu} = (m, -1, \ldots, -1)$

In this case:

$$\begin{array}{ll} K(q) & |\eta_1| \le e^{mq},\ |\eta_i| \le e^{-q}\ (i = 2, 3, \ldots, n) \\ \Lambda & \text{to consist of points} \\ & (x, \xi_1 x - y_1,\ \ldots,\ \xi_m x - y_m),\ (x, y_1, \ldots, y_m) \in \mathbb{Z}^n \end{array}$$

Note: (i) Lattice points lie in $K(q)$ if

$$(**) \qquad |x| \le e^{mq} \ \text{ and } \ |\xi_i x - y_i| \le e^{-q} \ \ (i = 1, 2, \ldots, m).$$

(ii)

$$\begin{array}{ll} & \mathrm{vol}(K) = 2^n \ \text{ and } \ \mathrm{covol}(\Lambda) = 1 \\ \Longrightarrow & \text{exists a lattice point in } K(q) \ \ \text{(Dirichlet).} \end{array}$$

Divide $(**)$ by x to get

$$\left| \xi_i - \frac{y_i}{x} \right| \le \frac{1}{x^{1+\frac{1}{m}}}, \quad (i = 1, 2, \ldots, m)$$

(Simultaneous approximation to $\xi_1, \ldots, \xi_m$).

Improve Dirichlet ?

$$\begin{aligned}
\alpha(p) \quad & \text{largest, so there exist } x, y_1, \ldots, y_m \text{ such that} \\
& |x| \leq e^p, \ |\xi_i x - y_i| \leq e^{-\alpha(p)p} \quad (i = 1, 2, \ldots, n) \\
& \alpha(p) \ \geq \ \frac{1}{m} \\
\alpha \quad & := \liminf \alpha(p) \\
\omega \quad & := \limsup \alpha(p)
\end{aligned}$$

Some other notation (α, ω), $(\widehat{\omega}, \omega)$, (α, β)

When $n = 1$, then $\alpha = 1$.

ω irrationality exponent of ξ:

$\omega = 1$ for $\xi = e$ and for ξ algebraic (Roth)
$\omega = ?$ for $\xi = \pi$ (cf. $\omega < 4$)

Dual quantities α^* and ω^* for "approximation to a linear form"

Jarnik, Khintchine (1920's, 1930's)
— Relations between $\alpha, \omega, \alpha^*, \omega^*$

$$\begin{aligned}(1+\alpha)(m+\overline{\varphi}_1) &= (1+\omega)(m+\underline{\varphi}_1) = n \\ (1+\alpha^*)(1-\underline{\varphi}_n) &= (1+\omega^*)(1-\overline{\varphi}_n) = n\end{aligned}$$

Jarnik $\quad n=3, \quad \overline{\varphi}_1+\underline{\varphi}_3+\overline{\varphi}_1\underline{\varphi}_3=0$

Khintchine $\quad (n-1)\underline{\varphi}_1+\overline{\varphi}_n \leq 0$ and $(n-1)\overline{\varphi}_n+\underline{\varphi}_1 \geq 0$

Further, Bugeaud, Laurent, Moshchevitin, German, S., S.

Conjecture 3.3. *Any inequality on $\underline{\varphi}_1, \overline{\varphi}_1, \ldots, \underline{\varphi}_n, \overline{\varphi}_n$ remains valid if $\underline{\varphi}_i \mapsto \overline{\varphi}_{n+1-i}$ and $\overline{\varphi}_i \mapsto \underline{\varphi}_{n+1-i}$ and the inequality is reversed.*