An Application of the Geometry of Numbers to Curves

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What is an “a-number” and why should I care?

The Cohen-Lenstra Conjectures posit “averages” of certain quantities associated with real quadratic fields (more precisely, with their class groups). The same philosophy applies equally well to function fields of hyperelliptic curves over finite fields. We are thus interested in the structure of the class group (i.e., Jacobian) of such curves (which we identify with their function fields). Among the relevant quantities attached to our curves/function fields (or the Jacobian/class group) is the a-number.
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Among the relevant quantities attached to our curves/function fields (or the Jacobian/class group) is the a-number.
Some Notation

A power of an odd prime $p$.

$F_q$ is the finite field with $q$ elements.

$X$ is transcendental over $F_q$, so that $F_q(X)$ is a field of rational functions $f(X) \in F_q[X]$ a monic square-free polynomial of degree 2 $g + 2$ ($g \geq 1$).

$E$ is the hyperelliptic curve of genus $g$ given by $Y^2 = f(X)$.

We associate with the function field obtained by adjoining $Y$ to the field of rational functions $F_q(X)$.

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In particular, $E_f$ is called “ordinary” if $a(E_f) = 0$. 
Previous Work

Cais, Ellenberg, and Zureik-Brown in 2013 studied "random $p$-divisible groups" and considered the question of whether such objects captured relevant features of random hyperelliptic Jacobians. Their evidence pointed to a negative answer, at least in some cases. Specifically, in the case of characteristic $p = 3$ computational evidence indicates that the probability that a random hyperelliptic curve over $\mathbb{F}_q$ is ordinary is $1 - q^{-1}$, whereas the random $p$-divisible group heuristic would suggest that this probability is

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(1 - q^{-1})(1 - q^{-3})(1 - q^{-5}) \cdots
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Moreover, they didn’t hazard a guess for higher $a$-numbers.
Elkin in 2011 did some estimates for a-numbers of Kummer covers of $\mathbb{P}^1_k$. For the special case of characteristic $p = 3$ he obtains the bound $a(E_f) < g_f^3 + 2$ where $g_f$ is the genus of the curve $E_f$. 
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$$a(E_f) < \frac{g_f}{3} + 2$$

where $g_f$ is the genus of the curve $E_f$. 
Characteristic 3

Here we’ll look specifically at the case where $p = 3$. In this case, everything revolves around solutions to the homogeneous linear equation (over $\mathbb{F}_q(X)$)

$$f_0 P_1 + f_1 P_2 + f_2 P_3 = 0$$

(1)

The “coefficient vector” $(f_0, f_1, f_2) \in \mathbb{F}_q[X]^3$ will correspond to our polynomial $f$ above.

The “solutions” $(P_1, P_2, P_3)$ will correspond to elements of the nullspace of $C_f$. 

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Write

\[ f(X) = \sum_{i \leq 2g+2} c_i X^i + X \cdot \sum_{i \leq 2g+2} c_i X^{i-1} + X^2 \cdot \sum_{i \leq 2g+2} c_i X^{i-2}. \]
The Coefficient Vector

Write

\[ f(X) = \sum_{i \leq 2g+2 \atop i \equiv 0 \mod 3} c_i X^i + X \cdot \sum_{i \leq 2g+2 \atop i \equiv 1 \mod 3} c_i X^{i-1} + X^2 \cdot \sum_{i \leq 2g+2 \atop i \equiv 2 \mod 3} c_i X^{i-2}. \]

Then

\[ f_j(X)^3 = \sum_{i \leq 2g+2 \atop i \equiv j \mod 3} c_i X^{i-j} \quad j = 0, 1, 2. \]
The Coefficient Vector

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\[ f(X) = \sum_{i \leq 2g+2, \quad i \equiv 0 \mod 3} c_i X^i + X \cdot \sum_{i \leq 2g+2, \quad i \equiv 1 \mod 3} c_i X^{i-1} + X^2 \cdot \sum_{i \leq 2g+2, \quad i \equiv 2 \mod 3} c_i X^{i-2}. \]

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The degrees of the \( f_j \)s are constrained depending on the congruence class of \( g \) modulo 3.
Constraints on the Degrees

$\frac{g-1}{3} = m$

For $g \equiv 0 \pmod{3}$:

$\frac{2g}{3} = 2m + 2 = \deg(f_2) \geq \max\{\deg(f_0), \deg(f_1)\}$

For $g \equiv 2 \pmod{3}$:

$\frac{2g+2}{3} = 2m + 2 = \deg(f_0) > \max\{\deg(f_1), \deg(f_2)\}$

For $g \equiv 1 \pmod{3}$:

$\frac{2g+1}{3} = 2m + 1 = \deg(f_1) \geq \deg(f_0), \deg(f_1) > \deg(f_2)$

Note also that $f$ is cube-free if and only if $f_0, f_1, f_2$ are relatively prime.
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Set $m = [(g - 1)/3]$. 

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Solutions and the Nullspace of $C_f$

Set $\omega = dX$. $C_f$ acts on the $\mathbb{F}_q$-vector space with basis $\{\omega, X\omega, \ldots, X^{g-1}\omega\}$ via $C_f(X_i\omega) = X^{[i/3]}f^{[2i+2]} \equiv j \mod 3$.

Thus elements of the nullspace of $C_f$ correspond to solutions $(P_1, P_2, P_3)$ to (1) with certain degree restrictions depending on the congruence class of $g$ modulo 3.
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Thus elements of the nullspace of $C_f$ correspond to solutions $(P_1, P_2, P_3)$ to (1) with certain degree restrictions depending on the congruence class of $g$ modulo 3.
We only consider polynomial solutions $P_1, P_2, P_3$ to $f_0 P_1 + f_1 P_2 + f_2 P_3 = 0$ satisfying $\deg(P_3) \leq m$, $\deg(P_2) \leq \{m \text{ if } g \not\equiv 1 \mod 3, m - 1 \text{ if } g \equiv 1 \mod 3\}$, and $\deg(P_1) \leq \{m \text{ if } g \not\equiv 1 \mod 3, 2 \text{ if } g \equiv 1 \mod 3\}$. 

An Application of the Geometry of Numbers to Curves
Degree Restrictions on Solutions

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$$\deg(P_1) \leq \begin{cases} m & \text{if } g \not\equiv 1, 2 \mod 3, \\ m - 1 & \text{if } g \equiv 1, 2 \mod 3. \end{cases}$$
Suppose $g \equiv 0 \mod 3$, so that $m = (g - 3)/3$. We see that $E$ has positive $a$-number if and only if there is a solution $(P_1, P_2, P_3)$ to (1) of height no greater than $m$. However, the solution space itself has height $2m + 2$, so we're asking for an abnormally small solution.

If there is a solution of height $l \leq m$, then all our sought-after solutions are projectively equivalent and $a(E) = m + 1 - l$. 

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The $a$-Number and Heights

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As noted above, we also have additional conditions beyond just the height for the coefficient vector of (1).

Nevertheless, we are able to compute exactly the number of cube-free $f$ of the necessary shape which would have a given $a(E_f)$ if it were also square-free.
Our Results

Suppose \( g \equiv 0 \mod 3 \) and positive. For any \( a \in \{0, \ldots, g/3\} \) the proportion \( P_g(a) \) of cube-free \( f \) which would have \( a(E_f) = a \) if they were square-free is exactly:

\[
P_g(a) = q - 2 \frac{g}{3} q,
\]

\[
P_g(a) = q - 2 a (q - q - 1) \quad \text{if} \quad 0 < a < g/3,
\]

\[
P_g(0) = 1 - q - 1.
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Moreover, our proportions for higher \( a \)-numbers also agree with the computational evidence compiled by Cais, Ellenberg, and Zureik-Brown.
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Our counting arguments don’t rely on characteristic 3 at all.
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Moreover, our proportions for higher \( a \)-numbers also agree with the computational evidence compiled by Cais, Ellenberg, and Zureik-Brown.

Our counting arguments don’t rely on characteristic 3 at all. But the connection between \( a \)-numbers and the analogous linear equation becomes more tenuous for larger characteristic.
Example of a Counting Result

For the case \( g \equiv 0 \mod 3 \), we use the following counting result.

Fix integers \( m \geq l \geq 0 \).

Let \( N_1(m, l) \) denote the number of ordered triples \((f_2, f_1, f_0)\) of relatively prime polynomials such that \( f_2 \) is monic of degree \( 2m + 2 \), the degrees of both \( f_1 \) and \( f_0 \) are no greater than \( 2m + 2 \), and there is a solution to (1) of height \( l \).

Then \( N_1(m, l) = \begin{cases} q^{4m - l} & \text{if } l > 0, \\ q^{4m}q^{5}(q^2 - 1)^2 & \text{if } l = 0. \end{cases} \)
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N_1(m, l) = \begin{cases} 
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$$N_1(m, l) = \begin{cases} q^{4m} - lq^{3} & \text{if } l > 0, \\ q^{4m}q^{5}(q^{2} - 1) & \text{if } l = 0. \end{cases}$$
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