Multiplicatively dependent vectors of algebraic numbers

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Diophantine Approximation and Analytic Number Theory in Honor of Jeffrey Vaaler

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Let n be a positive integer, G be a multiplicative group and let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ be in G^n. We say that $\boldsymbol{\nu}$ is multiplicatively dependent if there is a non-zero vector $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ for which

$$\boldsymbol{\nu}^{\mathbf{k}} = \nu_1^{k_1} \cdots \nu_n^{k_n} = 1. \tag{1}$$

We denote by $M_n(G)$ the set of multiplicatively dependent vectors in G^n.

For instance, the set $M_n(\mathbb{C}^*)$ of multiplicatively dependent vectors in $(\mathbb{C}^*)^n$ is of Lebesgue measure zero, since it is a countable union of sets of measure zero. Further, if we fix an exponent vector $\mathbf{k}$ the subvariety of $(\mathbb{C}^*)^n$ determined by (1) is an algebraic subgroup of $(\mathbb{C}^*)^n$.

Our objective is to count the number of multiplicatively dependent n-tuples whose coordinates are algebraic numbers of fixed degree, or within a fixed number field, and bounded height.

Equivalently we shall count n-tuples of algebraic numbers in a fixed algebraic number field, or of fixed degree, and given height which occur in some proper algebraic subgroup of the algebraic group G_m^n, where G_m is the multiplicative group of an algebraic closure of $\mathbb{Q}$.

For any algebraic number α, let

$$f(x) = a_d x^d + \cdots + a_1 x + a_0$$

be the minimal polynomial of α over the integers $\mathbb{Z}$ (so with content 1 and positive leading coefficient). Suppose that f factors as

$$f(x) = a_d(x - \alpha_1) \cdots (x - \alpha_d)$$

over the complex numbers $\mathbb{C}$. The *naive height* $H_0(\alpha)$ of α is given by

$$H_0(\alpha) = \max\{|a_d|, \ldots, |a_1|, |a_0|\},$$

and $H(\alpha)$, the height of α, also known as the *absolute Weil height* of α, is defined by

$$H(\alpha) = (a_d \prod_{i=1}^{d} \max\{1, |\alpha_i|\})^{1/d}.$$

Let K be a number field of degree d (over $\mathbb{Q}$). We use the following standard notation:

- r_1 and r_2 for the number of real and pairs of complex conjugate embeddings of K, respectively, and put $r = r_1 + r_2 - 1$;
- D, h, R and ζ_K for the discriminant, class number, regulator and Dedekind zeta function of K, respectively;
- w for the number of roots of unity in K.

Note that r is exactly the rank of the unit group of the ring of algebraic integers of K. As usual, let $\zeta(s)$ be the Riemann zeta function.

For any real number x, let $\lceil x \rceil$ denote the smallest integer greater than or equal to x, and let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

For a finite set S we use $|S|$ to denote its cardinality.

Let K be a number field of degree d. Denote the set of algebraic integers of K of height at most H by $\mathcal{B}_K(H)$ and the set of algebraic numbers of K of height at most H by $\mathcal{B}_K^*(H)$. Set

$$B_K(H) = |\mathcal{B}_K(H)| \qquad \text{and} \qquad B_K^*(H) = |\mathcal{B}_K^*(H)| \, .$$

Put

$$C_1(K) = \frac{2^{r_1}(2\pi)^{r_2}d^r}{|D|^{1/2}r!}.$$

Widmer(2016) proved that

$$B_K(H) = C_1(K)H^d(\log H)^r + O(H^d(\log H)^{r-1}). \qquad (2)$$

For any positive integer n, we denote by $L_{n,K}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic integers of height at most H, and we denote by $L^*_{n,K}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic numbers of height at most H.

Put

$$C_3(n, K) = \frac{n(n+1)}{2} w C_1(K)^{n-1}.$$

Theorem

Let K be a number field of degree d over $\mathbb{Q}$ and let n be an integer with $n \geq 2$. We have

$$L_{n,K}(H) = C_3(n,K)H^{d(n-1)}(\log H)^{r(n-1)} + O\left(H^{d(n-1)}(\log H)^{r(n-1)-1}\right); \tag{3}$$

if furthermore $K = \mathbb{Q}$ or is an imaginary quadratic field, we have

$$L_{n,K}(H) = C_3(n,K)H^{d(n-1)} + O\left(H^{d(n-3/2)}\right). \tag{4}$$

Define

$$C_2(K) = \frac{2^{2r_1}(2\pi)^{2r_2}2^r hR}{|D|w\zeta_K(2)}.$$

Schanuel proved in 1979 that

$$B_K^*(H) = C_2(K)H^{2d} + O(H^{2d-1}(\log H)^{\sigma(d)}), \qquad (5)$$

where $\sigma(1) = 1$ and $\sigma(d) = 0$ for $d > 1$.

We estimate $L^*_{n,K}(H)$ next. Put

$$C_4(n,K) = n^2 w C_2(K)^{n-1}.$$

Theorem

Let K be a number field of degree d, and let n be an integer with $n \geq 2$. Then, we have

$$L^*_{n,K}(H) = C_4(n,K)H^{2d(n-1)} + O(H^{2d(n-1)-1}g(H)), \qquad (6)$$

where

$$g(H) = \begin{cases} \log H & \text{if } d = 1 \text{ and } n = 2 \\ \exp(c \log H / \log\log H) & \text{if } d = 1 \text{ and } n > 2 \\ 1 & \text{if } d > 1 \text{ and } n \geq 2, \end{cases}$$

and c is a positive number depending only on n.

The following notion plays a crucial role in our argument. Let $\overline{\mathbb{Q}}$ be an algebraic closure of the rational numbers $\mathbb{Q}$. For each $\boldsymbol{\nu}$ in $(\overline{\mathbb{Q}}^*)^n$, we define s, the *multiplicative rank* of $\boldsymbol{\nu}$, in the following way. If $\boldsymbol{\nu}$ has a coordinate which is a root of unity, we put $s = 0$; otherwise let s be the largest integer with $1 \leq s \leq n$ for which any s coordinates of $\boldsymbol{\nu}$ form a multiplicatively independent vector. Notice that

$$0 \leq s \leq n - 1, \quad (7)$$

whenever $\boldsymbol{\nu}$ is multiplicatively dependent.

We now outline the strategy of the proofs. Given a number field K, we define $L_{n,K,s}(H)$ and $L^*_{n,K,s}(H)$ to be the number of multiplicatively dependent n-tuples of multiplicative rank s whose coordinates are algebraic integers in $\mathcal{B}_K(H)$ and algebraic numbers in $\mathcal{B}^*_K(H)$ respectively. It follows from (7) that

$$\begin{cases} L_{n,K}(H) = L_{n,K,0}(H) + \cdots + L_{n,K,n-1}(H) \\ L^*_{n,K}(H) = L^*_{n,K,0}(H) + \cdots + L^*_{n,K,n-1}(H). \end{cases} \tag{8}$$

The main term in (3) comes from the contributions of $L_{n,K,0}(H)$ and $L_{n,K,1}(H)$ in (8), and the main term in our second theorem comes from the contributions of $L^*_{n,K,0}(H)$ and $L^*_{n,K,1}(H)$ in (8). To prove our results we make use of (8) and the following result.

Theorem

Let K be a number field of degree d. Let n and s be integers with $n \geq 2$ and $0 \leq s \leq n-1$. Then, there exist positive numbers c_1 and c_2 which depend on n and K, such that

$$L_{n,K,s}(H) < H^{d(n-1)-d(\lceil (s+1)/2 \rceil -1)} \exp(c_1 \log H / \log\log H) \quad (9)$$

and

$$L^*_{n,K,s}(H) < H^{2d(n-1)-d(\lceil (s+1)/2 \rceil -1)} \exp(c_2 \log H / \log\log H). \quad (10)$$

The next result shows that if algebraic numbers $\alpha_1, \ldots, \alpha_n$ are multiplicatively dependent, then we can find a relation where the exponents are not too large. Such a result has found application in transcendence theory.

Lemma

Let $n \geq 2$, and let $\alpha_1, \ldots, \alpha_n$ be multiplicatively dependent non-zero algebraic numbers of degree at most d and height at most H. Then, there is a positive number c, which depends only on n and d, and there are rational integers $k_1, \ldots, k_n$, not all zero, such that

$$\alpha_1^{k_1} \cdots \alpha_n^{k_n} = 1$$

and

$$\max_{1 \leq i \leq n} |k_i| < c(\log H)^{n-1}.$$

This follows from a result of van der Poorten and Loxton.

Let x and y be positive real numbers with y larger than 2, and let $\psi(x, y)$ denote the number of positive integers not exceeding x which contain no prime factors greater than y. Put

$$Z = \left(\log\left(1 + \frac{y}{\log x}\right)\right)\frac{\log x}{\log y} + \left(\log\left(1 + \frac{\log x}{y}\right)\right)\frac{y}{\log y}$$

and

$$u = (\log x)/(\log y).$$

Lemma

For $2 < y \leq x$*, we have*

$$\psi(x,y) = \exp\left(Z\left(1 + O((\log y)^{-1}) + O((\log\log x)^{-1}) + O((u+1)^{-1})\right)\right)$$

This is a result of N.G. de Bruijn from 1966.

Let d be a positive integer, and let $\mathcal{A}_d(H)$, respectively $\mathcal{A}_d^*(H)$, be the set of algebraic integers of degree d (over $\mathbb{Q}$), respectively algebraic numbers of degree d, of height at most H. We set

$$A_d(H) = |\mathcal{A}_d(H)| \qquad \text{and} \qquad A_d^*(H) = |\mathcal{A}_d^*(H)|\,.$$

Put

$$C_5(d) = d2^d \prod_{j=1}^{\lfloor (d-1)/2 \rfloor} \frac{d(2j)^{d-2j-1}}{(2j+1)^{d-2j}}$$

and

$$C_6(d) = \frac{d2^d}{\zeta(d+1)} \prod_{j=1}^{\lfloor (d-1)/2 \rfloor} \frac{(d+1)(2j)^{d-2j}}{(2j+1)^{d-2j+1}}.$$

It follows from the work of Barroero from 2014 that

$$A_d(H) = C_5(d)H^{d^2} + O\left(H^{d(d-1)}(\log H)^{\rho(d)}\right), \qquad (11)$$

where $\rho(2) = 1$ and $\rho(d) = 0$ for any $d \neq 2$.

Masser and Vaaler showed in 2008 that

$$A_d^*(H) = C_6(d)H^{d(d+1)} + O\left(H^{d^2}(\log H)^{\vartheta(d)}\right), \qquad (12)$$

where $\vartheta(1) = \vartheta(2) = 1$ and $\vartheta(d) = 0$ for any $d \geq 3$.

For any positive integer n, we denote by $M_{n,d}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic integers in $\mathcal{A}_d(H)$, and we denote by $M^*_{n,d}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic numbers in $\mathcal{A}^*_d(H)$.

For each positive integer d, we define $w_0(d)$ to be the number of roots of unity of degree d. Let φ denote Euler's totient function. Since $\varphi(k) \gg k/\log\log k$ for any integer $k \geq 3$, it follows that

$$w_0(d) \ll d^2 \log\log d, \tag{13}$$

where $d \geq 3$ and the implied constant is absolute. We remark that $w_0(d)$ can be zero, such as for an odd integer $d > 1$.

Given positive integers n and d, we define $C_7(n,d)$ and $C_8(n,d)$ as

$$C_7(n,d) = (nw_0(d) + n(n-1))\, C_5(d)^{n-1}$$

and

$$C_8(n,d) = (nw_0(d) + 2n(n-1))\, C_6(d)^{n-1}.$$

Theorem

Let d and n be positive integers with $n \geq 2$. Then, the following hold.

(i) *We have*

$$M_{n,d}(H) = C_7(n,d)H^{d^2(n-1)} + O\left(H^{d^2(n-1)-d/2}\right); \tag{14}$$

furthermore if $d = 2$ or d is odd, we have

$$M_{n,d}(H) = C_7(n,d)H^{d^2(n-1)} + O\left(H^{d^2(n-1)-d}\exp(c_0 \log H / \log\log H)\right) \tag{15}$$

Theorem

Let d and n be positive integers with $n \geq 2$. Then, the following hold.

(ii) *We have*

$$M^*_{n,d}(H) = C_8(n,d)H^{d(d+1)(n-1)} + O\left(H^{d(d+1)(n-1)-d/2}\log H\right); \quad (16)$$

furthermore if $d = 2$ or d is odd, we have

$$M^*_{n,d}(H) = C_8(n,d)H^{d(d+1)(n-1)} + O\left(H^{d(d+1)(n-1)-d}\exp(c\log H/\log\log H)\right) \quad (17)$$

and where c is a positive number which depends only on n and d.

Thank you.