Height Bounds Under Splitting Conditions

AMS Special Session on Diophantine Approximation and Analytic Number Theory in Honor of Jeffrey Vaaler

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Let $\alpha$ be an algebraic number, $\alpha_1, \ldots, \alpha_n$ its Galois conjugates. The **Weil height** of $\alpha$ is the quantity

$$h(\alpha) = \sum_{p \leq \infty} \frac{1}{n} \sum_{i=1}^{n} \log^+ |\alpha_i|_p.$$  

Basic examples: $h(a/b) = \log \max\{|a|, |b|\}$ for rational $a/b$. 

Golden ratio: $h\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1}{2} \log \frac{1+\sqrt{5}}{2}$.

- The height measures size or complexity.
- $h(\alpha^n) = |n| \cdot h(\alpha)$ and $h(\sigma \alpha) = h(\alpha)$.
- For $f$ rational of degree $d$, $h(f(\alpha)) = d \cdot h(\alpha) + O_f(1)$.

Motivating question for this talk: If we impose conditions that the algebraic numbers must satisfy, how is the height affected?
The Notion of Height

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An algebraic number $\alpha$ is **totally real** if it satisfies any of the equivalent conditions:

- Galois conjugates $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{C}$ all lie in $\mathbb{R}$.
- The minimal polynomial of $\alpha$ over $\mathbb{Q}$ splits completely over $\mathbb{R}$.
- The infinite prime of $\mathbb{Q}(\alpha)$ splits completely.
- Every embedding $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ lies in $\mathbb{R}$.

**Theorem (Schinzel, 1973)**

If $\alpha \neq 0, \pm 1$ is totally real, then

$$h(\alpha) \geq \frac{1}{2} \log \frac{1 + \sqrt{5}}{2} > 0.$$
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Likewise, we say an algebraic number $\alpha$ is \textit{totally $p$-adic} for the prime $p$ if it satisfies any of the equivalent conditions:

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\textbf{Theorem (Bombieri, Zannier, 2001)}

$$\liminf_{\alpha \in \mathbb{Q}_{\text{tot.}}^p} h(\alpha) \geq \frac{1}{2} \frac{\log p}{p + 1}.$$
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Let $\alpha$ be an algebraic number, with conjugates $\alpha_1, \cdots, \alpha_n$. Define

$$[\alpha] = \frac{1}{n} \sum_{i=1}^{n} \delta_{\alpha_i}.$$

Theorem (Equidistribution for small height - Bilu 1997; Baker-Rumely, Favre-Rivera-Letelier 2006)

For any sequence of distinct algebraic numbers $(\alpha_n)$ with $h(\alpha_n) \to 0$, we have that the measures $[\alpha_n]$ converge weakly to the equilibrium distribution of the unit disc in $\mathbb{C}_p$ in the right context.
Conjugates of $\alpha = \zeta_{43} + \zeta_{43}^{-1}$, $h(\alpha) = 0.31442796\ldots$.

The conjugates of a totally real number cannot equidistribute around $S^1$, so the height cannot be arbitrarily small.
Suppose for the moment that

$$\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{Z}_p \subset \mathbb{Q}_p.$$ 

$\mathbb{Z}_p$ is a “small” subset of the unit disc $D \subset \mathbb{C}_p$, as $\mathbb{C}_p/\mathbb{Q}_p$ is an infinite dimensional extension. It turns out that the size of $\mathbb{Z}_p$ determines the height bound—if we think in terms of energy.
Reasons for the these bounds

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For a compactly supported Borel measure $\mu$ on the plane $\mathbb{C}_p$, we define the energy integral to be

$$I(\mu) = \int\int -\log |x - y|_p \, d\mu(x) \, d\mu(y).$$

Frostman’s theorem tells us that for any compact $K \subset \mathbb{C}_p$ there exists a unique measure $\mu_K$ with minimal energy.

The capacity of $K$ is $\text{cap}(K) = e^{-I(\mu_K)}$.

$\mathbb{Z}_p$ has capacity $p^{-1/(p-1)} < 1 = \text{cap}(D)$, that is,

$$I(\mu_{\mathbb{Z}_p}) = \frac{\log p}{p - 1}.$$
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Recall the height is $h(\alpha) = \sum_{p \leq \infty} \frac{1}{n} \sum_{i=1}^{n} \log^+ |\alpha|_p$. The $\log^+ |x|_p$ factor happens to be the potential associated to the unit disc in $\mathbb{C}_p$.

**Theorem (F., 2013)**

If $\alpha$ is a totally $p$-adic integer at the finite rational primes $p \in S$, then

$$h(\alpha) \geq \frac{1}{2} \sum_{p \in S} \frac{\log p}{p - 1} + o(1)$$

where the little-$O$ constant depends on the degree of $\alpha$.

Limits of this approach: As we replace $\mathbb{Z}_p$ by $\mathbb{Q}_p$, the capacity is infinite.
Energy integrals and capacity

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Limits of this approach: As we replace \( \mathbb{Z}_p \) by \( \mathbb{Q}_p \), the capacity is infinite.
Add to $-\log |x - y|_p$ kernel the external field $\log^+ |\cdot|$: 

$$- \log \delta_p(x, y) = \log^+ |x|_p + \log^+ |y|_p - \log |x - y|_p.$$ 

Key observation: 

$$h(\alpha) = \sum_{p \leq \infty} \frac{1}{n} \sum_{i=1}^{n} \log^+ |\alpha_i|_p = \frac{1}{2} \sum_{p \leq \infty} \frac{1}{n(n-1)} \sum_{i \neq j} - \log \delta_p(\alpha_i, \alpha_j).$$
Height and energy with external fields

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This leads us to consider, for a local field \( L \) and a probability measure \( \mu \) on \( L \), the \( \delta \)-energy integral

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I_\delta(\mu) = \iint_{\mathbb{P}^1(L)^2} - \log \delta(x, y) \, d\mu(x) \, d\mu(y).
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Notice \( 2h(\alpha) \) is a ‘discrete’ analogue of \( I_\delta([\alpha]) \). Simeonov (2005) proved the existence and uniqueness of a measure minimizing the energy for external fields of this type on \( \mathbb{C} \).

For \( \mathbb{C} \) and \( \mathbb{C}_p \), the minimal \( \delta \)-energy is 0. The equilibrium measure is the equilibrium measure of the unit disc at each place.
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Let $L$ be a non-complex local field. If $L$ is non-archimedean, the equilibrium measure is the unique $\text{PGL}_2(O_L)$-invariant measure $\mu_L$ on $L$, and the energy is

$$I_\delta(\mu_L) = \frac{q \log q}{q^2 - 1} \left( \text{compare to B-Z: } \frac{\log q}{q + 1} \right)$$

where $q$ is the cardinality of the residue field of $L$. If $L = \mathbb{R}$, then the equilibrium measure $\mu_\mathbb{R}$ is given by

$$d\mu_\mathbb{R}(x) = \frac{1}{\pi^2 x} \log \left| \frac{x + 1}{x - 1} \right| \, dx$$

and the minimal energy is $I_\delta(\mu) = 7\zeta(3)/2\pi^2$. 
These local minimal energies result in the following height bound:

**Theorem (F., Petsche 2013)**

Let $S$ be a set of rational primes, and $L_S$ be the field of all numbers totally $p$-adic for finite $p \in S$ and totally real if $\infty \in S$. Then

$$\lim \inf_{\alpha \in L_S} h(\alpha) \geq \frac{1}{2} \sum_{\substack{p \in S \atop p \mid \infty}} \frac{p \log p}{p^2 - 1} + \sum_{\substack{p \in S \atop p = \infty}} \frac{7 \zeta(3)}{4 \pi^2}$$

Further, if any of these bounds are achieved for some sequence of $\alpha_n$, then the measures $[\alpha_n]$ converge weakly to the minimal energy measures $\mu_p$ at each place (equidistribution).

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What about for numbers of a fixed degree?

The above theorem is ineffective in the sense that it depends on the degree of the algebraic numbers tending to infinity. Can we make the dependence on the degree explicit?

Problem

One cannot work directly with the discrete measures $[\alpha]$ in potential theory, as they do not have finite energy and do not have a continuous potential. So we cannot compute $I_\delta([\alpha])$, nor can we say that $h(\alpha) \geq \frac{1}{2} I_\delta(\mu_\mathbb{R})$ directly even when $\text{supp}([\alpha]) \subset \mathbb{R}$ and $\mu_\mathbb{R}$ yields the minimal energy on $\mathbb{R}$.
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We can regularize our discrete in the following fashion: Fix $\epsilon > 0$, and let

$$[\beta]_\epsilon = \frac{1}{n} \sum_{i=1}^{n} \delta_{\beta_i, \epsilon}$$

where $\delta_{\beta_i, \epsilon}$ is the normalized arc length measure on the circle $|z - \beta_i| = \epsilon$. These measures admit a continuous potential, and minimize certain associated quantities.
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Plot of $[\alpha]_{\epsilon}$

The energy $I_\delta([\alpha]_{\epsilon})$ is now finite!

But we cannot compare $I_\delta([\alpha]_{\epsilon})$ to $I(\mu_\mathbb{R})$, because supp($[\alpha]_{\epsilon}$) $\not\subset \mathbb{R}$.
Regularization process

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How can we do this?
We prove for the $\delta$-energy a common result from potential: given any finite energy measure $\mu$ supported on $E$, the minimal energy $V_\delta(E)$ satisfies

$$\inf_{z \in E} U_\mu(z) \leq V_\delta(E) \leq \sup_{z \in E} U_\mu(z).$$

So we can study the potential

$$U_{\mu_{\mathbb{R}}}(z) = \int -\log \delta(x, y) \, d\mu_{\mathbb{R}}(x) \, d\mu_{\mathbb{R}}(y)$$

in a neighborhood of the real line.
Some Results

Theorem (F.-Pottmeyer, 2017)

Let $E$ be as above for $0 < \epsilon < 1$. The $\delta$-Robin constant of $E_\epsilon$ satisfies

$$V_\delta(E_\epsilon) \geq I(\mu_\mathbb{R}) - 0.95\sqrt{\epsilon} = \frac{7\zeta(3)}{2\pi^2} - 0.95\sqrt{\epsilon}.$$

An analogous result is be proven, using the Berkovich space theorem, in the $p$-adic context. This leads us to a more explicit version of the original F.-Petsche result.
Theorem (F.-Pottmeyer, 2017)

Let $\alpha$ be totally $p$-adic for all $p \in S$ (totally real if $\infty \in S$ of degree $d = [\mathbb{Q}(\alpha) : \mathbb{Q}] > 1$. Set

$$V_\infty = \begin{cases} 0 & \text{if } \infty \notin S \\ \max \left\{ \frac{7\zeta(3)}{4\pi^2} - \frac{0.95d+2}{2d^2} - \frac{(d-2)\log d}{2d(d-1)}, 0 \right\} & \text{if } \infty \in S \end{cases}$$

Then we have

$$h(\alpha) \geq -\frac{\log d}{2(d - 1)} + V_\infty$$

$$+ \frac{1}{2} \sum_{p \in S, \ p \neq \infty, \ p < d} \left( \left(1 - \frac{1}{p^{n_p-1}}\right) \frac{p \log p}{p^2 - 1} - \frac{\log d}{d} \right)$$

where for each finite $p$, $n_p = \left\lfloor \frac{\log d}{\log p} \right\rfloor$. 
Thanks to the organizers for putting this session together, to Jeff for the inspiration and years of support, and to you for listening!