Height Bounds Under Splitting Conditions

AMS Special Session on Diophantine Approximation and Analytic Number Theory in Honor of Jeffrey Vaaler

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The Notion of Height

Let α be an algebraic number, $\alpha_1, \dots, \alpha_n$ its Galois conjugates. The Weil height of α is the quantity

$$h(\alpha) = \sum_{p \le \infty} \frac{1}{n} \sum_{i=1}^{n} \log^+ |\alpha_i|_p .$$

Basic examples: $h(a/b) = \log \max\{|a|, |b|\}$ for rational a/b.
Golden ratio: $h\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1}{2} \log \frac{1+\sqrt{5}}{2}$.

- The height measures size or complexity.
- $h(\alpha^n) = |n| \cdot h(\alpha)$ and $h(\sigma\alpha) = h(\alpha)$.
- For f rational of degree d, $h(f(\alpha)) = d \cdot h(\alpha) + O_f(1)$.

Motivating question for this talk: If we impose conditions that the algebraic numbers must satisfy, how is the height affected?

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First condition: Totally real

An algebraic number α is totally real if it satisfies any of the equivalent conditions:

- Galois conjugates $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{C}$ all lie in $\mathbb{R}$.
- The minimal polynomial of α over $\mathbb{Q}$ splits completely over $\mathbb{R}$.
- The infinite prime of $\mathbb{Q}(\alpha)$ splits completely.
- Every embedding $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ lies in $\mathbb{R}$.

Theorem (Schinzel, 1973)

If $\alpha \neq 0, \pm 1$ is totally real, then

$$h(\alpha) \geq \frac{1}{2} \log \frac{1+\sqrt{5}}{2} > 0.$$

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Theorem (Bombieri, Zannier, 2001)

$$\liminf_{\alpha \in \mathbb{Q}^{tot.\ p}} h(\alpha) \geq \frac{1}{2} \frac{\log p}{p+1}.$$

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Equidistribution

Let α be an algebraic number, with conjugates $\alpha_1, \cdots, \alpha_n$. Define

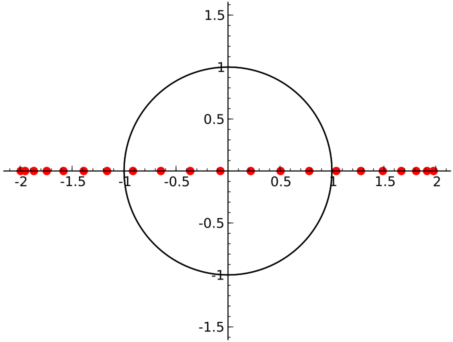
$$[\alpha] = \frac{1}{n}\sum_{i=1}^{n} \delta_{\alpha_i}.$$

Theorem (Equidistribution for small height - Bilu 1997; Baker-Rumely, Favre-Rivera-Letelier 2006)

For any sequence of distinct algebraic numbers (α_n) with $h(\alpha_n) \to 0$, we have that the measures $[\alpha_n]$ converge weakly to the equilibrium distribution of the unit disc in $\mathbb{C}_p$ in the right context.

A proof by picture of Schinzel's theorem

Conjugates of $\alpha = \zeta_{43} + \zeta_{43}^{-1}$, $h(\alpha) = 0.31442796\ldots$.



The conjugates of a totally real number cannot equidistribute around S^1, so the height cannot be arbitrarily small.

Reasons for the these bounds

Suppose for the moment that

$$\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{Z}_p \subset \mathbb{Q}_p.$$

$\mathbb{Z}_p$ is a "small" subset of the unit disc $D \subset \mathbb{C}_p$, as $\mathbb{C}_p/\mathbb{Q}_p$ is an infinite dimensional extension. It turns out that the size of $\mathbb{Z}_p$ determines the height bound—if we think in terms of energy.

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Energy integrals and capacity

For a compactly supported Borel measure μ on the plane $\mathbb{C}_p$, we define the energy integral to be

$$I(\mu) = \iint -\log|x-y|_p \, d\mu(x)\, d\mu(y).$$

Frostman's theorem tells us that for any compact $K \subset \mathbb{C}_p$ there exists a unique measure μ_K with minimal energy.

The capacity of K is $\mathrm{cap}(K) = e^{-I(\mu_K)}$.

$\mathbb{Z}_p$ has capacity $p^{-1/(p-1)} < 1 = \mathrm{cap}(D)$, that is,

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Theorem (F., 2013)

If α is a totally p-adic integer at the finite rational primes $p \in S$, then

$$h(\alpha) \ge \frac{1}{2} \sum_{p \in S} \frac{\log p}{p - 1} + o(1)$$

where the little-O constant depends on the degree of α.

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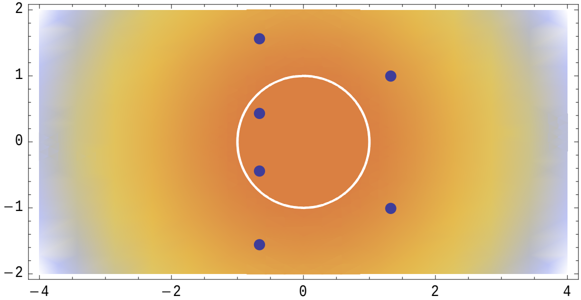
Height and energy with external fields

Add to $-\log |x-y|_p$ kernel the external field $\log^+ |\cdot|$:

$$-\log \delta_p(x,y) = \log^+ |x|_p + \log^+ |y|_p - \log |x-y|_p \,.$$



Key observation:

$$h(\alpha) = \sum_{p \le \infty} \frac{1}{n} \sum_{i=1}^{n} \log^+ |\alpha_i|_p = \frac{1}{2} \sum_{p \le \infty} \frac{1}{n(n-1)} \sum_{i \ne j} -\log \delta_p(\alpha_i, \alpha_j).$$

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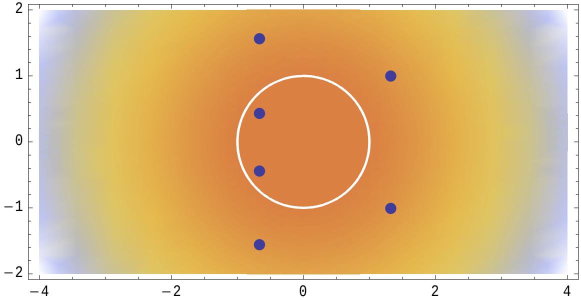

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Energies of subsets with this external field

This leads us to consider, for a local field L and a probability measure μ on L, the δ-energy integral

$$I_\delta(\mu) = \iint_{\mathbb{P}^1(L)^2} -\log \delta(x,y)\, d\mu(x)\, d\mu(y).$$

Notice $2h(\alpha)$ is a 'discrete' analogue of $I_\delta([\alpha])$. Simeonov (2005) proved the existence and uniqueness of a measure minimizing the energy for external fields of this type on $\mathbb{C}$.

For $\mathbb{C}$ and $\mathbb{C}_p$, the minimal δ-energy is 0. The equilibrium measure is the equilibrium measure of the unit disc at each place.

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Theorem (F., Petsche 2013)

Let L be a non-complex local field. If L is non-archimedean, the equilibrium measure is the unique $\mathrm{PGL}_2(O_L)$-invariant measure μ_L on L, and the energy is

$$I_\delta(\mu_L) = \frac{q \log q}{q^2 - 1} \quad \left(\text{compare to B-Z: } \frac{\log q}{q+1}\right)$$

where q is the cardinality of the residue field of L. If $L = \mathbb{R}$, then the equilibrium measure $\mu_{\mathbb{R}}$ is given by

$$d\mu_{\mathbb{R}}(x) = \frac{1}{\pi^2 x} \log \left| \frac{x+1}{x-1} \right| dx$$

and the minimal energy is $I_\delta(\mu) = 7\zeta(3)/2\pi^2$.

Quantitative bounds arising from capacity theory

These local minimal energies result in the following height bound:

Theorem (F., Petsche 2013)

Let S be a set of rational primes, and L_S be the field of all numbers totally p-adic for finite $p \in S$ and totally real if $\infty \in S$. Then

$$\liminf_{\alpha \in L_S} h(\alpha) \geq \frac{1}{2} \sum_{\substack{p \in S \\ p \nmid \infty}} \frac{p \log p}{p^2 - 1} + \sum_{\substack{p \in S \\ p = \infty}} \frac{7\zeta(3)}{4\pi^2}$$

Further, if any of these bounds are achieved for some sequence of α_n, then the measures $[\alpha_n]$ converge weakly to the minimal energy measures μ_p at each place (equidistribution).

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What about for numbers of a fixed degree?

The above theorem is ineffective in the sense that it depends on the degree of the algebraic numbers tending to infinity. Can we make the dependence on the degree explicit?

Problem

One cannot work directly with the discrete measures $[\alpha]$ in potential theory, as they do not have finite energy and do not have a continuous potential. So we cannot compute $I_\delta([\alpha])$, nor can we say that $h(\alpha) \geq \frac{1}{2} I_\delta(\mu_{\mathbb{R}})$ directly even when $\mathrm{supp}([\alpha]) \subset \mathbb{R}$ and $\mu_{\mathbb{R}}$ yields the minimal energy on $\mathbb{R}$.

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Regularization of discrete measures

We can regularize our discrete in the following fashion: Fix $\epsilon > 0$, and let

$$[\beta]_\epsilon = \frac{1}{n} \sum_{i=1}^{n} \delta_{\beta_i,\epsilon}$$

where $\delta_{\beta_i,\epsilon}$ is the normalized arc length measure on the circle $|z - \beta_i| = \epsilon$. These measures admit a continuous potential, and minimize certain associated quantities.

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Regularization process

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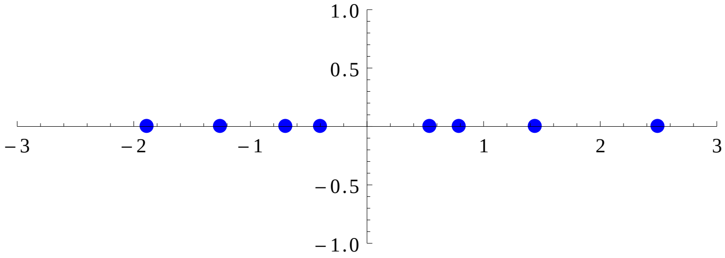
$f(x) = x^8 - x^7 - 7x^6 + 4x^5 + 13x^4 - 4x^3 - 7x^2 + x + 1.$



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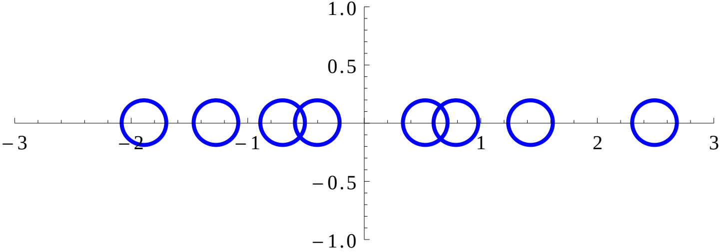

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The energy $I_\delta([\alpha]_\epsilon)$ is now finite!

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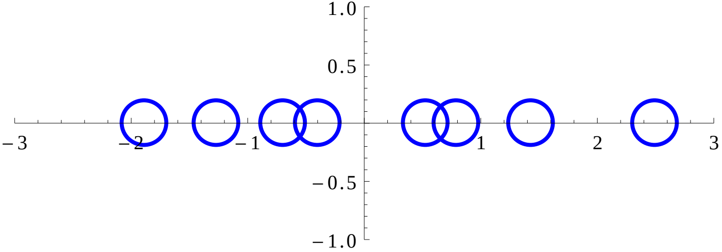

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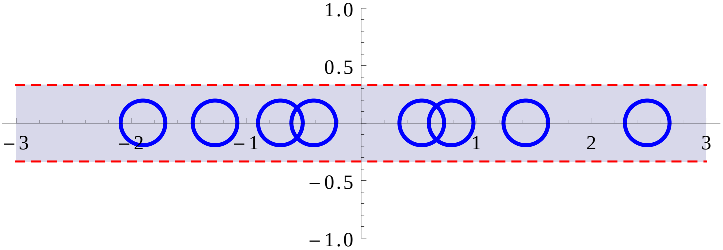

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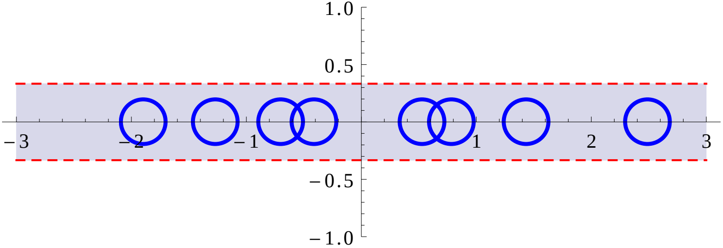

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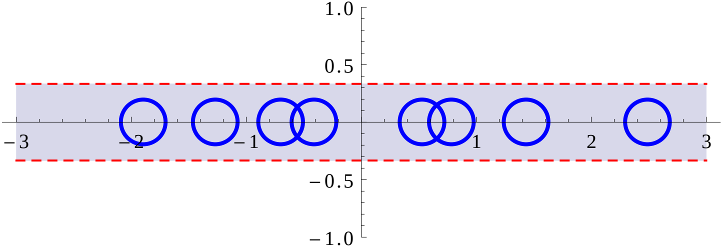

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Estimating minimal energies of ϵ-neighborhoods

We prove for the δ-energy a common result from potential: given any finite energy measure μ supported on E, the minimal energy $V_\delta(E)$ satisfies

$$\inf_{z\in E} U_\mu(z) \leq V_\delta(E) \leq \sup_{z\in E} U_\mu(z).$$

So we can study the potential

$$U_{\mu_{\mathbb{R}}}(z) = \int -\log \delta(x,y)\, d\mu_{\mathbb{R}}(x)\, d\mu_{\mathbb{R}}(y)$$

in a neighborhood of the real line.

Some Results

Theorem (F.-Pottmeyer, 2017)

Let E be as above for $0 < \epsilon < 1$. The δ-Robin constant of E_ϵ satisfies

$$V_\delta(E_\epsilon) \geq I(\mu_{\mathbb{R}}) - 0.95\sqrt{\epsilon} = \frac{7\zeta(3)}{2\pi^2} - 0.95\sqrt{\epsilon}.$$

An analogous result is be proven, using the Berkovich space theorem, in the p-adic context. This leads us to a more explicit version of the original F.-Petsche result.

Some Results

Theorem (F.-Pottmeyer, 2017)

Let α be totally p-adic for all $p \in S$ (totally real if $\infty \in S$ of degree $d = [\mathbb{Q}(\alpha) : \mathbb{Q}] > 1$. Set

$$V_\infty = \begin{cases} 0 & \text{if } \infty \notin S \\ \max\left\{ \frac{7\zeta(3)}{4\pi^2} - \frac{0.95d+2}{2d^2} - \frac{(d-2)\log d}{2d(d-1)}, 0 \right\} & \text{if } \infty \in S \end{cases}$$

Then we have

$$h(\alpha) \geq -\frac{\log d}{2(d-1)} + V_\infty + \frac{1}{2} \sum_{\substack{p \in S \\ p \neq \infty,\ p<d}} \left(\left(1 - \frac{1}{p^{n_p - 1}}\right) \frac{p \log p}{p^2 - 1} - \frac{\log d}{d} \right)$$

where for each finite p, $n_p = \left\lfloor \frac{\log d}{\log p} \right\rfloor$.

Thanks

Thanks to the organizers for putting this session together, to Jeff for the inspiration and years of support, and to you for listening!