AMS Special Session in honor of Jeff Vaaler
Joint Mathematical Meetings, San Diego

New Trigonometric Extremal Problems Related to Pair Correlations

Hugh L. Montgomery
University of Michigan
Ann Arbor, MI

Friday, January 12, 2018

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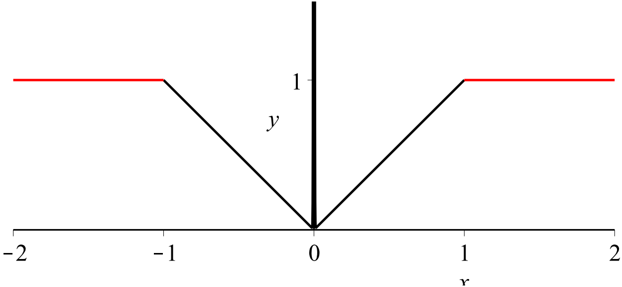
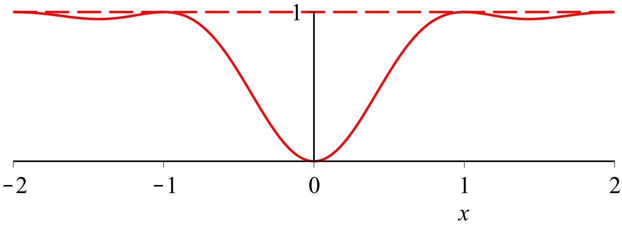


$F(\alpha)$

Conjectural Pair Correlation



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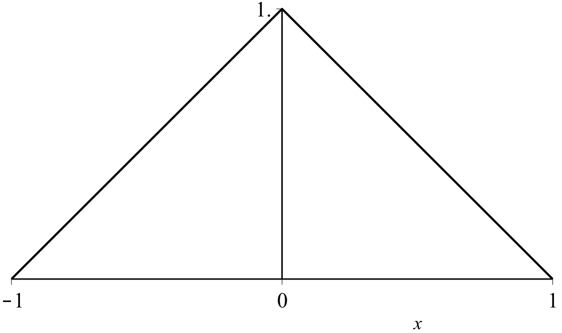

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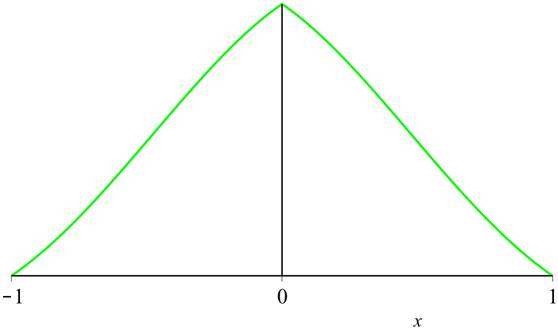

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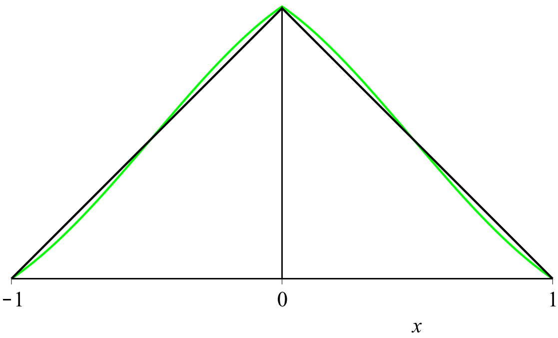

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Try $\widehat{r}$ of the form

$$a\exp(-b\alpha^2)(1-\alpha^2) \qquad \text{or} \qquad a\exp(-b|\alpha|)(1-\alpha^2)$$

What are the possible F?

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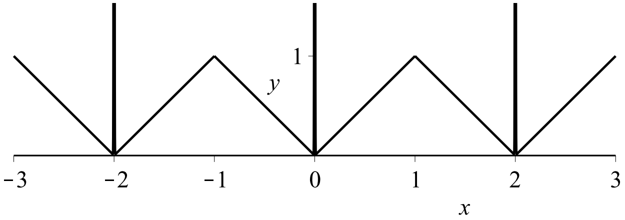
The Alternative Hypothesis:



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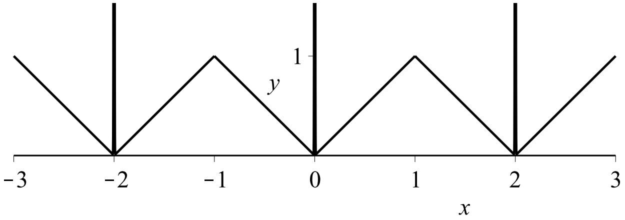

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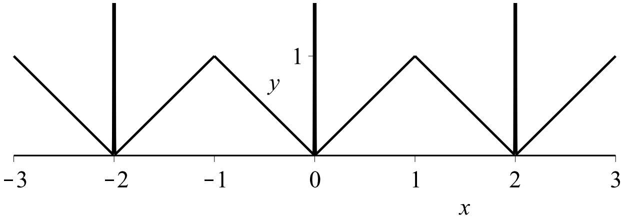

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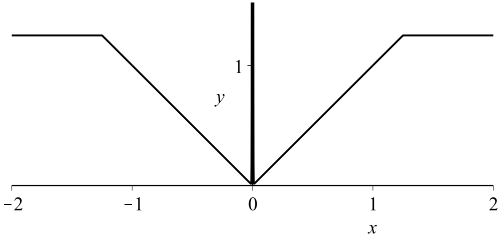

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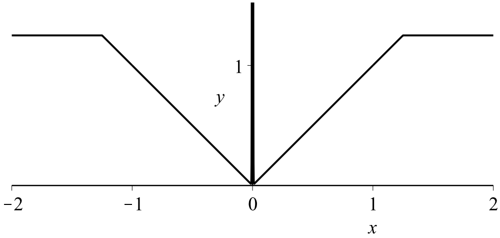

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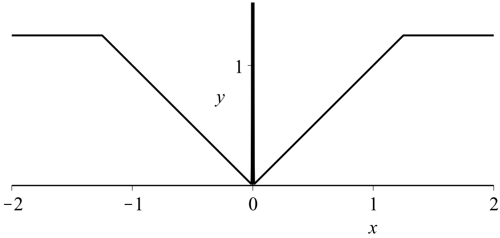

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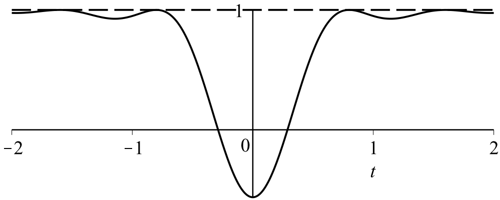
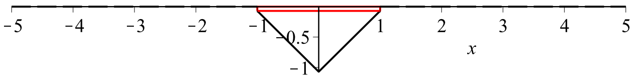
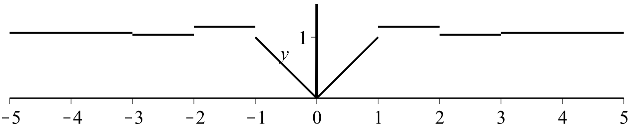



−5
−4
−3
−2
−1
1
2
3
4
5
x
−0.5
−1

Possible F



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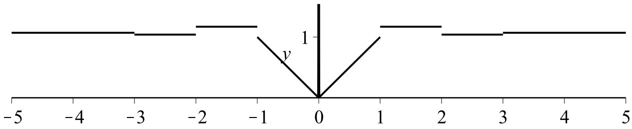

Asymptote $= 1.07$

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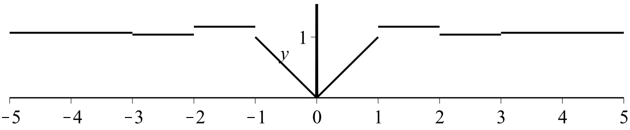

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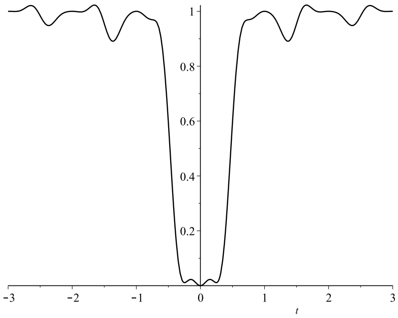

Close Pairs of Zeros

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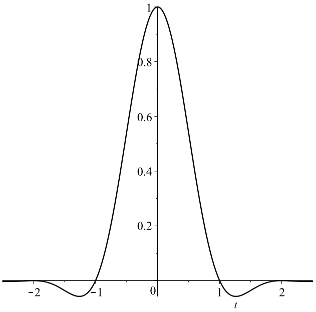
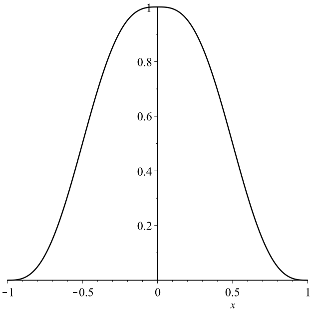


$f(x)$



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