Iteration In The Ideal Sieve

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I knew immediately what he was talking about. Jeff wanted an amazing function that majorized e^{-x} for positive x and 0 for negative x. This eventually led to our joint paper on extremal functions.

Definitions and background for the Ideal Sieve

Start with a finite set of integers $\mathcal{A}$.
Each $n \in \mathcal{A}$ is equipped with a non-negative weight w_n.

Let $\mathcal{P}$ be a finite set of primes and

$$P = \prod_{p \in \mathcal{P}} p.$$

The sieve problem is to obtain upper and lower bounds for

$$S_1 = \sum_{\substack{n \in \mathcal{A} \\ (n,P)=1}} w_n,$$

from information about

$$W_d = \sum_{\substack{n \in \mathcal{A} \\ d|n}} w_n.$$

Typically, estimates for W_d take the form

$$g(d)X - R_d^- \leq W_d \leq g(d)X + R_d^+$$

for $d|P$, where g is a non-negative multiplicative function.

A set of coefficients $\{\lambda_d : d|P\}$ is an *upper bound sieve* if for every $n \in \mathcal{A}$,

$$\sum_{d|(n,P)} \lambda_d \geq \begin{cases} 1 & \text{if } (n,P) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For an upper bound sieve,

$$S_1 \leq \sum_{n \in \mathcal{A}} w_n \sum_{d|(n,P)} \lambda_d \leq X \sum_{d|P} g(d)\lambda_d + \sum_{d|P} |\lambda_d| R_d^{\operatorname{sgn} \lambda_d}.$$

A similar construction gives lower bound sieves–just replace $\leq$ with $\geq$.

In "Lectures on Sieves" Selberg discussed "a simple sifting problem" In this problem, the primes in $\mathcal{P}$ satisfy

$$z^{1/(R+1)} \leq p < z^{1/R}$$

for arbitrary integer $R \geq 1$.

This is sometimes referred to as "Selberg's Toy Sieve."

We will look at the same problem, but with some extra machinery.

The Ideal Sieve

Error terms are usually controlled by requiring $\lambda_d = 0$ for $d \geq z$, where z is some appropriate parameter.
Montgomery suggested the following idealization: Assume

$$R_d = \begin{cases} 0 & \text{if } d < z \\ \infty & \text{if } d \geq z \end{cases}$$

Therefore, the only useful sieves have $\lambda_d = 0$ for $d \geq z$.
If $d < z$, then

$$W_d = \sum_{\substack{n \in \mathcal{A} \\ d|n}} w_n = g(d)X$$

By homogenity, we may normalize to $X = 1$.

Looking for Extremal Examples

The key idea: Make a change of basis. Recall

$$W_d = \sum_{\substack{n \in \mathcal{A} \\ d|n}} w_n = g(d).$$

Define

$$S_d = \sum_{\substack{n \in \mathcal{A} \\ (n,P)=d}} w_n.$$

We can recover W_d from S_d via Möbius inversion:

$$W_d = \sum_{e | \frac{P}{d}} S_{de}, \quad S_d = \sum_{e | \frac{P}{d}} \mu(e) W_{de}.$$

Henceforth, we describe sets in terms of S_d instead of W_d or w_n.

Admissible Sets

The set $\{S_d : d|P\}$ is admissible if

1. $S_d \geq 0$ for all $d|P$, and
2. if $d|P$ and $d \leq z$, then $\displaystyle\sum_{d|\delta|P} S_\delta = g(d)$.

If $\{\lambda_d\}$ is an upper bound sieve, then for any $\mathcal{A}$,

$$S_1 \leq \sum_{n\in\mathcal{A}} w_n \sum_{d|n} \lambda_d = \sum_{d|P} \lambda_d g(d).$$

If we can find an admissible set $\{S_d\}$ such that

$$S_1 = \sum_{d|P} \lambda_d g(d),$$

then $\{\lambda_d\}$ is an optimal upper bound sieve.
A corresponding construction works for lower bounds

Define $\theta_m = \sum_{d|m} \lambda_d$. The condition for an upper bound sieve may be rephrased as

$$\theta_1 \geq 1, \quad \theta_m \geq 0 \text{ for } m > 1.$$

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In our examples, θ_m will depend only on $\omega(m)$. If $\omega(m) = \ell$, we write

$$\theta(\ell) \text{ in place of } \theta_m.$$

Our Basic Problem

As in Selberg's Toy Sieve, we assume that the sifting primes p lie in the range

$$z^{1/(R+1)} < p \leq z^{1/R}.$$

We wish to identify best possible upper and lower bound sieves, and identify extremal g.

A useful observation

Lemma

Suppose λ_d is an optimal upper bound sieve and $\{S_d\}$ is a corresponding admissible set. If $d > 1$, then $S_d = 0$ or $\theta_d = 0$.

Proof.

$$S_1 = \sum_{d|P} \lambda_d g(d) = \sum_{d|P} \lambda_d \sum_{r|\frac{P}{d}} S_{dr}$$

$$= \sum_{m|P} S_m \sum_{d|m} \lambda_d = \sum_{m|P} S_m \theta_m.$$

$\theta_m \geq 0$ and $\theta_1 \geq 1$, we must have $S_m \theta_m = 0$ for $m > 1$. □

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However, it's a useful case to look at.

$R = 1$: Sifting primes in $(z^{1/2}, z]$

Say that the sieving primes are $p_1, \ldots, p_K$ with

$$z^{1/2} < p_i \leq z.$$

The product of any two of these primes exceeds z, so $\lambda_d = 0$ if $\omega(d) \geq 2$.

For the lower bound, we take $\theta(n) = 1 - \omega(n)$. Then

$$\lambda_1 = 1, \lambda_p = -1.$$

Note that λ_d depends only on the number of prime factors of d, so it is convenient to write

$$\lambda(0) = 1 \text{ and } \lambda(1) = -1.$$

Therefore

$$S_1 = S(0) \geq 1 - Kg.$$

The extremal S must have $S(\ell) = 0$ when $\ell \geq 2$.

Moreover, the admissibility conditions are equivalent to the system of equations

$$\begin{aligned} S(0) + KS(1) &= 1 \\ S(1) &= g. \end{aligned}$$

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Note that $S(0)$ is equal to the sieve bound above, so our choice of θ is best possible.

When $Kg > 1$, the best lower bound is the trivial estimate $S_1 = 0$. However, it is useful generalize the previous construction. Assume that r is an integer and $r \le Kg \le r + 1$. Assume that $S(\ell) = 0$ unless $\ell = r$ or $\ell = r + 1$. Then the admissibility conditions become

$$\binom{K}{r} S(r) + \binom{K}{r+1} S(r+1) = 1$$
$$\binom{K-1}{r-1} S(r) + \binom{K-1}{r} S(r+1) = g. \tag{1}$$

This system has solution

$$S(r) = \frac{r + 1 - Kg}{\binom{K}{r}}, \quad S(r+1) = \frac{Kg - r}{\binom{K}{r+1}}.$$

By our hypotheses, both $S(r)$ and $S(r+1)$ are non-negative.

$R = 2$: Optimal upper bound sieve when $z^{1/3} < p \le z^{1/2}$

In this case, $\lambda_d = 0$ if $\omega(d) \ge 3$.

Write $\theta_d = \sum_{e|d} \lambda_e = \theta(\ell)$ when $\omega(d) = \ell$.

We need $\theta(0) \geq 1, \theta(\ell) \geq 0$. Take

$$\theta(\ell) = \left(1 - \frac{\ell}{r}\right)\left(1 - \frac{\ell}{r+1}\right).$$

where r is a positive integer to be chosen later.

Upper Bound for $R = 2$: $z^{1/3} < p \leq z^{1/2}$

We take

$$\theta(\ell) = \left(1 - \frac{\ell}{r}\right)\left(1 - \frac{\ell}{r+1}\right).$$

The admissibility equations are

$$\begin{aligned}
\binom{K}{0}S(0) + \binom{K}{r}S(r) + \binom{K}{r+1}S(r+1) &= 1, \\
\binom{K-1}{r-1}S(r) + \binom{K-1}{r}S(r+1) &= g, \\
\binom{K-2}{r-2}S(r) + \binom{K-2}{r-1}S(r+1) &= g^2,
\end{aligned}$$

The solution is

$$\begin{aligned} S(0) =& 1 - \frac{2}{r+1} K g + \frac{2}{r(r+1)} \binom{K}{2} g^2, \\ S(r) =& \frac{(r - (K-1)g)g}{\binom{K-1}{r-1}}, \\ S(r+1) =& \frac{((K-1)g - r + 1)g}{\binom{K-1}{r}}. \end{aligned}$$

so we have an admissible system provided

$$(K-1)g \le r \le (K-1)g + 1. \tag{2}$$

Thus we may take $r = [(K-1)g]$.

A Comparison

With $R = 1$, we have

$$S(r) = \frac{r + 1 - Kg}{\binom{K}{r}}, \quad S(r+1) = \frac{Kg - r}{\binom{K}{r+1}}.$$

With $R = 2$, we have

$$S(r) = \frac{(r - (K-1)g)g}{\binom{K-1}{r-1}}, \quad S(r+1) = \frac{((K-1)g - r + 1)g}{\binom{K-1}{r}}.$$

Note that we can go from the first to the second by making the substitutions

$$r \to r - 1, \qquad K \to K - 1$$

and multiplying by g.

With $R = 3$ and the lower bound sieve

$$\theta(\ell) = (1-\ell)(1-\frac{\ell}{r})(1-\frac{\ell}{r+1})$$

we get

$$S(r) = \frac{(r-1-(K-2)g)g^2}{\binom{K-2}{r-2}}, \quad S(r+1) = \frac{((K-2)g-r+2)g}{\binom{K-2}{r-1}}.$$

Again, this same as $R = 2$ with $r \to r-1$ and $K \to K-1$.

Summary

Going to the admissibility equations, we see that the substitutions $r \to r-1$ and $K \to K-1$ take the admissibility equations for $R=1$ to admissibility equations for $R=2$. Similar phenomenon work for $R=2$ to $R=3$, et cetera.

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I don't what it looks like yet.

The End ...

Thank you! . . .