Fourier optimization and prime gaps

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(joint with M. Milinovich and K. Soundararajan)

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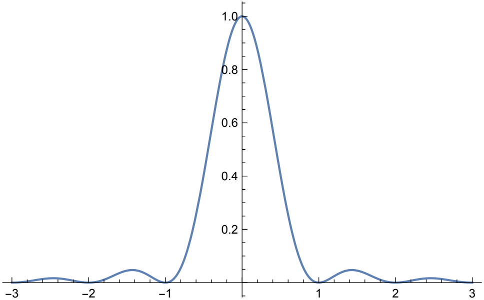
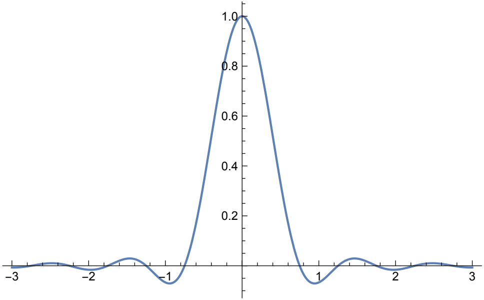
A classical problem

1. For $f : \mathbb{R} \to \mathbb{R}$, our normalization for Fourier transform is
$$\widehat{f}(t) = \int_{\mathbb{R}} e^{-2\pi ixt} f(x)\, \mathrm{d}x.$$
2. Let $F : \mathbb{R} \to \mathbb{R}$ be a nonnegative function, with $F(0) = 1$ and $\mathrm{supp}(\widehat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?
3. Answer =1.
$F(x) = (\sin(\pi x)/(\pi x))^2$



An 'innocent' variant

1. Let $F : \mathbb{R} \to \mathbb{R}$ be such that $F(0) = 1$ and $\mathrm{supp}(\widehat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?
2. $H(x) = \frac{\cos(2\pi x)}{1-16x^2}$ yields $\|H\|_{L^1(\mathbb{R})} = 0.9259...$



3. Best up-to-date (Gorbachev '05)

$$0.911 < C < 0.9243$$

4. There exists a unique extremizer.

Prime gaps

1. Bertrand's postulate (1845): is there always a prime in the interval $[x, 2x]$?

2. Tchebyshev (1852): Yes. There is always a prime in $[x, x + \frac{x}{\log x}]$ for x large.

3. Hoheisel (1930): There is always a prime in $[x, x + x^{\theta}]$ for some $0 < \theta < 1$, and x large.

4. Baker - Harman - Pintz (2001): There is always a prime in $[x, x + x^{0.525}]$ for x large.

Prime gaps on RH

Cramér's bounds (1920)

1. $$p_{n+1} - p_n = O(\sqrt{p_n}\log p_n),$$
   i.e. every interval $[x, x + c\sqrt{x}\log x]$, for some $c > 0$, contains a prime when x is large.

2. Historic progress:
   - Goldston '83: $c = 4$.
   - Ramaré and Saouter '03: $c = 8/5$
   - Dudek '15: $c = 1 + o(1)$.

3. Non-asymptotic version - best result due to Dudek, Grenié, Molteni '16: for $x \geq 4$, every interval
   $$[x, x + c\sqrt{x}\log x]$$
   contains a prime. Here $c = 1 + \frac{4}{\log x}$.
   - Ramaré and Saouter '03 ($c = 8/5$)

Recent progress

joint with M. Milinovich and K. Soundararajan

Theorem (Asymptotic version)

Assume RH. For x large, every interval

$$\left[x, x + \frac{21}{25}\sqrt{x}\log x\right]$$

contains a prime.

Theorem (Non-asymptotic version)

Assume RH. For $x \geq 4$, every interval

$$\left[x, x + \frac{22}{25}\sqrt{x}\log x\right]$$

contains a prime.

Strategy

1. Explicit formulas connecting zeros of $\zeta(s)$ and primes.
2. Brun-Titchmarsh inequality.
3. Fourier optimization problems.

Using the explicit formula

Let h be a function with mild smoothness/decay properties:

$$\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\widehat{h}(0)\log \pi$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\, \Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) \mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right),$$

where $\rho = \beta + i\gamma$ are the non-trivial zeros of $\zeta(s)$.

- Assume that an interval $[x, x + c\sqrt{x}\log x]$ has no primes.
- Let F be a test function. Idea is to use the formula with

$$h(z) = \Delta F(\Delta z)\, a^{iz}$$

  with $[x, x + c\sqrt{x}\log x] = [a\, e^{-2\pi\Delta}, a\, e^{2\pi\Delta}]$.
- Perform an asymptotic analysis (as $\Delta \to 0$ and $a \to \infty$).

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- Perform an asymptotic analysis (as $\Delta \to 0$ and $a \to \infty$).

This leads to...

1. If $\operatorname{supp}(\widehat{F}) \subset [-1,1]$ we would have
$$c \leq \frac{\|F\|_1}{F(0)}.$$
2. One can actually do better by (over)estimating in $[-1,1]^c$:
$$c \leq \frac{\|F\|_1}{\left(F(0) - \mathbf{B}\int_{[-1,1]^c} \left(\widehat{F}(t)\right)_+ \mathrm{d}t\right)}.$$
3. Here **B** is the Brun-Titchmarsh constant in our desired scale
$$\mathbf{B} := \limsup_{x\to\infty} \frac{\pi(x+\sqrt{x}) - \pi(x)}{\sqrt{x}/\log x}.$$
4. By the PNT (on the left) and work of Iwaniec (on the right):
$$1 \leq \mathbf{B} \leq \frac{36}{11}.$$

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Extremal problems

1. **Problem 1:** Given $1 \leq A \leq \infty$, find

$$\mathcal{C}(A) := \sup_{\substack{F \in \mathcal{A} \\ F \neq 0}} \frac{1}{\|F\|_1} \left(|F(0)| - A \int_{[-1,1]^c} \big|\widehat{F}(t)\big| \, \mathrm{d}t \right),$$

where the supremum is taken over the class $\mathcal{A}$ of continuous functions $F : \mathbb{R} \to \mathbb{C}$, with $F \in L^1(\mathbb{R})$.

2. **Problem 2:** Given $1 \leq A \leq \infty$, find

$$\mathcal{C}^+(A) := \sup_{\substack{F \in \mathcal{A}^+ \\ F \neq 0}} \frac{1}{\|F\|_1} \left(F(0) - A \int_{[-1,1]^c} \big(\widehat{F}(t)\big)_+ \, \mathrm{d}t \right),$$

where the supremum is taken over the class $\mathcal{A}^+$ of even and continuous functions $F : \mathbb{R} \to \mathbb{R}$, with $F \in L^1(\mathbb{R})$.

What we can prove

1. Existence of extremizers for $A > 1$. No extremizers for $A = 1$.
2. Uniqueness in the bandlimited problem (+ variational condition).
3. Good upper and lower bounds for all of these problems.
4. $A \mapsto \mathcal{C}(A)$ and $A \mapsto \mathcal{C}^+(A)$ are monotone decreasing with

$$2 = \mathcal{C}^+(1) = \mathcal{C}(1) \geq \mathcal{C}^+(A) \geq \mathcal{C}(A) \geq \mathcal{C}^+(\infty) \geq \mathcal{C}(\infty) \geq 1.0799...$$

5. $$\mathcal{C}^+\left(\frac{36}{11}\right) > \frac{25}{21} = 1.1948...$$

$$F(x) = -4.8\,x^2 e^{-3.3x^2} + 1.5\,x^2 e^{-7.4x^2} + 520\,x^{24} e^{-9.7x^2} + 1.3\,e^{-2.8x^2} + 0.18\,e^{-2x^2}$$

Conclusion 1

Theorem (Asymptotic version)

Assume RH. Then

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\sqrt{p_n}\log p_n} \leq \frac{1}{\mathcal{C}^+(\mathbf{B})} \leq \frac{1}{\mathcal{C}^+(36/11)} \leq \frac{21}{25}.$$

Conclusion 2 - non-asymptotic version

1. Use version of the Brun-Titchmarsh inequality due to Montgomery and Vaughan.

$$\pi(x+y) - \pi(x) < \frac{2y}{\log y},$$

for all $x, y > 1$. Relevant range is $y \sim \sqrt{x}$, which corresponds to $A = 4$ in the extremal problem.

2. With $H(x) = \frac{\cos(2\pi x)}{1-16x^2}$ and $\lambda = 0.9$ we use $F(x) = H(x/\lambda)$. Then

$$J(F) = \frac{F(0) - 4\int_{[-1,1]^c} \left|\widehat{F}(t)\right| \mathrm{d}t}{\|F\|_1} = 1.1405... > \frac{25}{22}.$$

3. Work the previous argument to make all error terms effective (Mellin transform approach slightly simpler).

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