Fourier optimization and prime gaps

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(joint with M. Milinovich and K. Soundararajan)

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A classical problem

1. For $f : \mathbb{R} \to \mathbb{R}$, our normalization for Fourier transform is

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i t x} f(x) \, dx.$$ 

2. Let $F : \mathbb{R} \to \mathbb{R}$ be a nonnegative function, with $F(0) = 1$ and $\text{supp}(\hat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?

3. Answer = 1.

$F(x) = \left( \frac{\sin(\pi x)}{\pi x} \right)^2$
An ‘innocent’ variant

1. Let $F : \mathbb{R} \to \mathbb{R}$ be such that $F(0) = 1$ and $\text{supp}(\hat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?

2. $H(x) = \frac{\cos(2\pi x)}{1-16x^2}$ yields $\|H\|_{L^1(\mathbb{R})} = 0.9259…$

3. Best up-to-date (Gorbachev ’05)

$$0.911 < C < 0.9243$$

4. There exists a unique extremizer.
Prime gaps

1. Bertrand’s postulate (1845): is there always a prime in the interval \([x, 2x]\)?

2. Tchebyshev (1852): Yes. There is always a prime in \([x, x + \frac{x}{\log x}]\) for \(x\) large.

3. Hoheisel (1930): There is always a prime in \([x, x + x^\theta]\) for some \(0 < \theta < 1\), and \(x\) large.

4. Baker - Harman - Pintz (2001): There is always a prime in \([x, x + x^{0.525}]\) for \(x\) large.
Prime gaps on RH
Cramér’s bounds (1920)

1. \[ p_{n+1} - p_n = O\left(\sqrt{p_n \log p_n}\right), \]
i.e. every interval \([x, x + c\sqrt{x} \log x]\), for some \(c > 0\), contains a prime when \(x\) is large.

2. Historic progress:
   - Goldston ’83: \(c = 4\).
   - Ramaré and Saouter ’03: \(c = 8/5\)
   - Dudek ’15: \(c = 1 + o(1)\).

3. Non-asymptotic version – best result due to Dudek, Grenié, Molteni ’16: for \(x \geq 4\), every interval
   \[ [x, x + c\sqrt{x} \log x] \]
   contains a prime. Here \(c = 1 + \frac{4}{\log x}\).
   - Ramaré and Saouter ’03 (\(c = 8/5\))
Recent progress
joint with M. Milinovich and K. Soundararajan

**Theorem (Asymptotic version)**

Assume RH. For $x$ large, every interval

$$[x, x + \frac{21}{25} \sqrt{x} \log x]$$

contains a prime.

**Theorem (Non-asymptotic version)**

Assume RH. For $x \geq 4$, every interval

$$[x, x + \frac{22}{25} \sqrt{x} \log x]$$

contains a prime.
1. Explicit formulas connecting zeros of $\zeta(s)$ and primes.

2. Brun-Titchmarsh inequality.

3. Fourier optimization problems.
Using the explicit formula

Let $h$ be a function with mild smoothness/decay properties:

\[
\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\hat{h}(0)\log\pi \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \Re \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{iu}{2}\right) \, du \\
- \frac{1}{2\pi} \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\hat{h}\left(\frac{\log n}{2\pi}\right) + \hat{h}\left(-\frac{\log n}{2\pi}\right)\right),
\]

where $\rho = \beta + i\gamma$ are the non-trivial zeros of $\zeta(s)$.

- Assume that an interval $[x, x + c\sqrt{x}\log x]$ has no primes.
- Let $F$ be a test function. Idea is to use the formula with

\[
h(z) = \Delta F(\Delta z) a^{iz}
\]

with $[x, x + c\sqrt{x}\log x] = [a e^{-2\pi\Delta}, a e^{2\pi\Delta}]$.
- Perform an asymptotic analysis (as $\Delta \to 0$ and $a \to \infty$).
Using the explicit formula

Let $h$ be a function with mild smoothness/decay properties:

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- Perform an asymptotic analysis (as $\Delta \to 0$ and $a \to \infty$).
This leads to...

1. If $\text{supp}(\hat{F}) \subset [-1, 1]$ we would have

   $$c \leq \frac{\|F\|_{1}}{F(0)}.$$

2. One can actually do better by (over)estimating in $[-1, 1]^c$:

   $$c \leq \frac{\|F\|_{1}}{\left(F(0) - B \int_{[-1,1]^c} (\hat{F}(t))_+ \, dt\right)}.$$

3. Here $B$ is the Brun-Titchmarsh constant in our desired scale:

   $$B := \limsup_{x \to \infty} \frac{\pi(x + \sqrt{x}) - \pi(x)}{\sqrt{x}/\log x}.$$

4. By the PNT (on the left) and work of Iwaniec (on the right):

   $$1 \leq B \leq \frac{36}{11}.$$
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   \[
   1 \leq B \leq \frac{36}{11}.
   \]
Problem 1: Given $1 \leq A \leq \infty$, find
\[ C(A) := \sup_{\substack{F \in A \\ F \neq 0}} \frac{1}{\|F\|_1} \left( |F(0)| - A \int_{[-1,1]^c} |\hat{F}(t)| \, dt \right), \]
where the supremum is taken over the class $A$ of continuous functions $F : \mathbb{R} \to \mathbb{C}$, with $F \in L^1(\mathbb{R})$.

Problem 2: Given $1 \leq A \leq \infty$, find
\[ C^+(A) := \sup_{\substack{F \in A^+ \\ F \neq 0}} \frac{1}{\|F\|_1} \left( F(0) - A \int_{[-1,1]^c} (\hat{F}(t))_+ \, dt \right), \]
where the supremum is taken over the class $A^+$ of even and continuous functions $F : \mathbb{R} \to \mathbb{R}$, with $F \in L^1(\mathbb{R})$. 

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What we can prove

2. Uniqueness in the bandlimited problem (+ variational condition).
3. Good upper and lower bounds for all of these problems.
4. $A \mapsto C(A)$ and $A \mapsto C^+(A)$ are monotone decreasing with

$$2 = C^+(1) = C(1) \geq C^+(A) \geq C(A) \geq C^+(\infty) \geq C(\infty) \geq 1.0799...$$

5. 

$$C^+ \left( \frac{36}{11} \right) > \frac{25}{21} = 1.1948...$$

$$F(x) = -4.8x^2e^{-3.3x^2} + 1.5x^2e^{-7.4x^2} + 520x^{24}e^{-9.7x^2} + 1.3e^{-2.8x^2} + 0.18e^{-2x^2}$$
Conclusion 1

**Theorem (Asymptotic version)**

Assume RH. Then

\[
\limsup_{n \to \infty} \frac{p_{n+1} - p_n}{\sqrt{p_n} \log p_n} \leq \frac{1}{c^+(B)} \leq \frac{1}{c^+(36/11)} \leq \frac{21}{25}.
\]
Use version of the Brun-Titchmarsh inequality due to Montgomery and Vaughan.

\[ \pi(x + y) - \pi(x) < \frac{2y}{\log y}, \]

for all \( x, y > 1 \). Relevant range is \( y \sim \sqrt{x} \), which corresponds to \( A = 4 \) in the extremal problem.

With \( H(x) = \frac{\cos(2\pi x)}{1 - 16x^2} \) and \( \lambda = 0.9 \) we use \( F(x) = H(x/\lambda) \). Then

\[ J(F) = \frac{F(0) - 4 \int_{[-1,1]} |\hat{F}(t)| dt}{\| F \|_1} = 1.1405... > \frac{25}{22}. \]

Work the previous argument to make all error terms effective (Mellin transform approach slightly simpler).