The Convolution Square Root of 1

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0. Sections of the talk

1. $1^{1/2}$, its convolution inverse $\mu^{1/2}$, and their properties

2. The PNT via $\sum 1^{1/2}$

3. Estimate of $\sum 1^{1/2}$

4. Suitability of our hypothesis
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2. The PNT via $\sum 1^{*1/2}$

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4. Suitability of our hypothesis

Spoiler alerts: a. $1^{*1/2} \neq \pm 1$. 
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2. The PNT via $\sum 1^{1/2}$

3. Estimate of $\sum 1^{1/2}$

4. Suitability of our hypothesis

Spoiler alerts:  

a. $1^{1/2} \neq \pm 1$.  

b. In the end (sigh) we use $\zeta(1 + it) \neq 0$. 
1 a. $1^{1/2}$ and its properties

The divisor function is $\tau = 1 \ast 1$, the multiplicative convolution square of the arithmetic function $1(n) = 1$, $\forall n$. 
1 a. $1^{*1/2}$ and its properties

The divisor function is $\tau = 1 * 1$, the multiplicative convolution square of the arithmetic function $1(n) = 1$, $\forall n$.

We study $1^{*1/2}$, the solution of $1^{*1/2} * 1^{*1/2} = 1$, $1^{*1/2}(1) = 1$. The generating function of $1^{*1/2}$ is the branch of $\zeta(s)^{1/2}$ that is positive on the real half-line $\{s > 1\}$. Also,

$$\zeta(s)^{1/2} = \prod_p \left(1 - p^{-s}\right)^{-1/2} = \prod_p \sum_{\nu=0}^{\infty} \left(-1/2\right)\nu (-1)^\nu p^{-\nu s}.$$ 

Thus $1^{*1/2}$ is a multiplicative function with

$$1^{*1/2}(p^\nu) = (-1)^\nu \binom{-1/2}{\nu} = \frac{1}{\nu!} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2\nu - 1}{2}$$ 

for $\nu \geq 1$, and so $1^{*1/2}$ is positive valued.
1 b. $\mu^{*1/2}$ and its properties

Let $\mu^{*1/2}$ be the convolution inverse of $1^{*1/2}$: $1^{*1/2} * \mu^{*1/2} = \delta$, where $\delta(1) = 1$ and $\delta(n) = 0$ else: the convolution identity.

The generating function of $\mu^{*1/2}$ is

$$\frac{1}{\zeta(s)^{1/2}} = \prod_p \left(1 - p^{-s}\right)^{1/2} = \prod_p \sum_{\nu=0}^{\infty} \left(\frac{1}{2}\right)(-1)^{\nu} p^{-\nu s}.$$ 

Thus $\mu^{*1/2}$ also is multiplicative, and since

$$\left|\binom{1/2}{\nu}\right| \leq \left|\binom{-1/2}{\nu}\right|, \quad \nu \geq 0,$$

we have $|\mu^{*1/2}(n)| \leq 1^{*1/2}(n)$ for all $n \geq 1$. 
2 a. Statement of PNT using \( \sum 1^{1/2} \) estimate

The condition

\[ N_{1/2}(x) := \sum_{n \leq x} 1^{1/2}(n) \sim x \log^{-1/2} x + cx \log^{-3/2} x \quad (1) \]

for some constant \( c \) implies the Prime Number Theorem.

(We do not need to know the value of \( c \) — it gets washed out.)
2 b. A Chebyshev type identity

Define operator $L$ on arithmetic functions: $Lf(n) = f(n) \log n$. $L$ is a derivation, so $L(f * f) = 2f * Lf$.

$\Lambda = \text{von Mangoldt's function}: \Lambda(p^\alpha) = \log p, \; \Lambda(n) = 0, \; \text{else.}$

Chebyshev's identity: $L1 = \Lambda * 1$.

Thus,

$$L1 = L(1^{1/2} * 1^{1/2}) = 2 \, 1^{1/2} * L1^{1/2} \bullet \Lambda * 1^{1/2} * 1^{1/2}.$$ 

and, convolving both sides of “$\bullet$” by $\mu^{1/2} * \mu^{1/2}$, we get

$$\Lambda = 2 \, L1^{1/2} * \mu^{1/2}.$$ 

Now sum, using the definition of convolution, to get

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{mn \leq x} 2 \, L1^{1/2}(m) \mu^{1/2}(n).$$
2. c. Idea for proving the PNT using $\sum 1^{1/2}$

Rewrite the $\psi$ identity as an iterated summation:

$$
\psi(x) = 2 \sum_{n \leq x} \left\{ \sum_{k \leq x/n} L1^{1/2}(k) \right\} \mu^{1/2}(n). \quad (2)
$$

Below, we shall apply Condition (1) to show that

$$
2 \sum_{k \leq x/n} L1^{1/2}(k) \approx \sum_{k \leq x/n} \left\{ 1 \ast 1^{1/2} + c'1^{1/2} \ast 1^{1/2} + c''1^{1/2} \right\}(k). \quad (3)
$$

Insert this into (2), use $1^{1/2} \ast \mu^{1/2} = \delta$, and we get

$$
\psi(x) = \sum_{n \leq x} \left( \left\{ 1 \ast 1^{1/2} + c'1^{1/2} \ast 1^{1/2} + c''1^{1/2} \right\} \ast \mu^{1/2} \right)(n) + o(x)
$$

$$
= \sum_{n \leq x} \left\{ 1(n) + c'1^{1/2}(n) + c''\delta(n) \right\} + o(x) \sim x.
$$
2 d. Discussion

The preceding method works if we have estimate (1) for $\sum 1^{1/2}$ and can give a suitable error analysis for $\sum L1^{1/2}$ in (3).

Why does this approach fail if we attempt to treat

$$\psi(x) = \sum_{n \leq x} \left\{ \sum_{m \leq x/n} L1(m) \right\} \mu(n)$$

by approximating $\sum L1$ by $\sum (1 * 1 + c1)$?

The problem here is that a priori we know only $\sum_{n \leq x} \mu(n) \ll x$, while for $\mu^{1/2}$ we have the better bound

$$\left| \sum_{n \leq x} \mu^{1/2}(n) \right| \leq \sum_{n \leq x} 1^{1/2}(n) \ll x \log^{-1/2} x.$$
2 e. Estimate of formula (3)

Assuming

\[ \sum_{n \leq x} 1^{*1/2}(n) = \frac{x}{\log^{1/2} x} + \frac{c x}{\log^{3/2} x} + o\left( \frac{x}{\log^{3/2} x} \right), \quad (1 \text{ bis}) \]

summing by parts yields

\[ \sum_{n \leq x} L1^{*1/2}(n) = x \log^{1/2} x + \frac{c'' x}{\log^{1/2} x} + o\left( \frac{x}{\log^{1/2} x} \right). \]

In place of \( \sum_{n \leq x}(1^{*1/2} \ast 1)(n) \), use the (equivalent) expression

\[ I := \int_{1}^{x} dN_{1/2} \ast dt = \int_{1}^{x} N_{1/2}(x/t) dt. \]

Using (1 bis) and easy estimates we find

\[ I = 2cx \log^{1/2} x + c'' x \frac{c'''}{\log^{1/2} x} + o\left( \frac{x}{\log^{1/2} x} \right). \]
2 f. Estimate of formula (3) continued

Also, trivially,

\[
\sum_{n \leq x} (1^{1/2} \ast 1^{1/2})(n) = \sum_{n \leq x} 1(n) = \lfloor x \rfloor,
\]

and

\[
\sum_{n \leq x} 1^{1/2}(n) = \frac{x}{\log^{1/2} x} + o\left(\frac{x}{\log^{1/2} x}\right).
\]

Thus, with suitable choices of \(c_2, c_3\), we have

\[
\Delta(x) := 2 \sum_{n \leq x} L1^{1/2}(n) - \int_1^x dN_{1/2} \ast dt - c_2 \lfloor x \rfloor - c_3 N_{1/2}(x)
= o\left(\frac{x}{\log^{1/2} x}\right).
\]
2 g. Conclusion of the proof of the PNT

From the formula for $\psi$, the estimate for $\Delta$, and the relation $1^{*1/2} \ast \mu^{*1/2} = \delta$ we find

$$\psi(x) = \sum_{n \leq x} (2 L 1^{*1/2} \ast \mu^{*1/2})(n) = 2 \sum_{n \leq x} \sum_{m \leq x/n} L 1^{*1/2}(m) \mu^{*1/2}(n)$$

$$= \int_1^x dt + c_2 N_{1/2}(x) + c_3 + E(x) = x + o(x) + E(x),$$

where

$$E(x) := \sum_{1}^{x} \Delta(x/n) \mu^{*1/2}(n) \ll \sum_{1}^{x} o\left(\frac{x/n}{\log^{1/2} x/n}\right) 1^{*1/2}(n).$$

Using the bound for $N_{1/2}$ again and making a simple estimate, we find $E(x) = o(x)$, and thus $\psi(x) \sim x$. 
3 a. Estimate of $\sum 1^{*1/2}$

Recall Dirichlet divisor function estimate

$$\sum_{n \leq x} 1^* 1(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

We establish

$$N_{1/2}(x) := \sum_{n \leq x} 1^{*1/2}(n) \sim \frac{x}{\log^{1/2} x} + \frac{cx}{\log^{3/2} x} \quad (1 \text{ bis})$$

using Perron inversion formula, assuming a modest zero-free region for $\zeta(s)$, the Riemann zeta function. For non-integral $x$,

$$N_{1/2}(x) = \frac{1}{2\pi i} \int_{C} x^s \zeta(s)^{1/2} ds / s, \quad (4)$$

with $C$ a vertical line to the right of 1 in $\mathbb{C}$. 
3 b. Estimate of $\sum 1^{1/2}$ continued

The main contribution to the integral arises from the half-order pole of $\zeta(s)^{1/2}$ at $s = 1$. We evaluate the integral by deforming the contour $\mathcal{C}$ to extend a bit to the left of the line $\{ \Re s = 1 \}$ but which encircles $s = 1$, and then using the Hankel loop integral formula. We find for non-integral $x$,

$$N_{1/2}(x) \sim x \log^{-1/2} x + (\gamma/2 - 1)x \log^{-3/2} x.$$  

Note that we used non-vanishing of $\zeta(s)$ and Mellin inversion, so, despite how it began, \textit{this proof of the PNT is not elementary}. 
4 a. Suitability of our hypothesis

Our proof of the PNT assumed that we know the first two terms of the asymptotic series for \( N_{1/2}(x) \). Here we indicate that this is an appropriate condition.

Suppose the PNT were false. Then, by familiar theory, \( \zeta(s) \) would have zeros at some points \( s = 1 \pm i\alpha \). Again use Perron’s formula

\[
N_{1/2}(x) = \frac{1}{2\pi i} \int_{C} x^s \zeta(s)^{1/2} ds / s 
\]  
(4 bis)

and take account of the half-order zeros. At \( 1 + i\alpha \), the Hankel loop integral formula gives an additional term \( ke^{i\theta} x^{1+i\alpha} \log^{-3/2} x \). This and the conjugate together give a contribution

\[
2kx \cos(\alpha \log x + \theta) \log^{-3/2} x,
\]

i.e. there would be a cosine wobble in the \( x \log^{-3/2} x \) term for \( N_{1/2}(x) \), so condition (1) could not hold.
4 b. A tauberian ingredient in our proof

Usual proofs of PNT contain a tauberian element. Here is ours:
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Usual proofs of PNT contain a tauberian element. Here is ours:

The formula

\[
\sum_{n \leq x} 1^{1/2} \ast 1^{1/2}(n) = \lfloor x \rfloor
\]

is an entangling of two copies of \(1^{1/2}\). Passage from \(1^{1/2} \ast 1^{1/2}\) to the asymptotic estimate

\[
\sum_{n \leq x} 1^{1/2}(n) \sim x \log^{-1/2} x + cx \log^{-3/2} x \quad (1 \text{ bis})
\]

is a tauberian activity.