The Convolution Square Root of 1

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0. Sections of the talk

1. $1^{*1/2}$, its convolution inverse $\mu^{*1/2}$, and their properties
2. The PNT via $\sum 1^{*1/2}$
3. Estimate of $\sum 1^{*1/2}$
4. Suitability of our hypothesis

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b. In the end (sigh) we use $\zeta(1 + it) \neq 0$.

1 a. $1^{*1/2}$ and its properties

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We study $1^{*1/2}$, the solution of $1^{*1/2} * 1^{*1/2} = 1, \quad 1^{*1/2}(1) = 1$. The generating function of $1^{*1/2}$ is the branch of $\zeta(s)^{1/2}$ that is positive on the real half-line $\{s > 1\}$. Also,

$$\zeta(s)^{1/2} = \prod_p \left(1 - p^{-s}\right)^{-1/2} = \prod_p \sum_{\nu=0}^{\infty} \binom{-1/2}{\nu} (-1)^\nu p^{-\nu s}.$$

Thus $1^{*1/2}$ is a multiplicative function with

$$1^{*1/2}(p^\nu) = (-1)^\nu \binom{-1/2}{\nu} = \frac{1}{\nu!} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2\nu - 1}{2} \text{ for } \nu \geq 1,$$

and so $1^{*1/2}$ is positive valued.

1 b. $\mu^{*1/2}$ and its properties

Let $\mu^{*1/2}$ be the convolution inverse of $1^{*1/2}$: $1^{*1/2} * \mu^{*1/2} = \delta$, where $\delta(1) = 1$ and $\delta(n) = 0$ else: the convolution identity.

The generating function of $\mu^{*1/2}$ is

$$\frac{1}{\zeta(s)^{1/2}} = \prod_p \left(1 - p^{-s}\right)^{1/2} = \prod_p \sum_{\nu=0}^{\infty} \binom{1/2}{\nu} (-1)^\nu p^{-\nu s}.$$

Thus $\mu^{*1/2}$ also is multiplicative, and since

$$\left| \binom{1/2}{\nu} \right| \le \left| \binom{-1/2}{\nu} \right|, \quad \nu \ge 0,$$

we have $|\mu^{*1/2}(n)| \le 1^{*1/2}(n)$ for all $n \ge 1$.

2 a. Statement of PNT using $\sum 1^{*1/2}$ estimate

The condition

$$N_{1/2}(x) := \sum_{n \le x} 1^{*1/2}(n) \sim x \log^{-1/2} x + cx \log^{-3/2} x \qquad (1)$$

for some constant c implies the Prime Number Theorem.

(We do not need to know the value of c — it gets washed out.)

2 b. A Chebyshev type identity

Define operator L on arithmetic functions: $\mathrm{L}f(n) = f(n)\log n$.
L is a derivation, so $\mathrm{L}(f * f) = 2f * \mathrm{L}f$.

Λ = von Mangoldt's function: $\Lambda(p^\alpha) = \log p$, $\Lambda(n) = 0$, else.

Chebyshev's identity: $\mathrm{L}1 = \Lambda * 1$.

Thus,

$$\mathrm{L}1 = \mathrm{L}(1^{*1/2} * 1^{*1/2}) = 2\,1^{*1/2} * \mathrm{L}1^{*1/2} \doteq \Lambda * 1^{*1/2} * 1^{*1/2}.$$

and, convolving both sides of "$\doteq$" by $\mu^{*1/2} * \mu^{*1/2}$, we get

$$\Lambda = 2\,\mathrm{L}1^{*1/2} * \mu^{*1/2}.$$

Now sum, using the definition of convolution, to get

$$\psi(x) := \sum_{n \le x} \Lambda(n) = \sum_{mn \le x} 2\,\mathrm{L}1^{*1/2}(m)\,\mu^{*1/2}(n).$$

2. c. Idea for proving the PNT using $\sum 1^{*1/2}$

Rewrite the ψ identity as an iterated summation:

$$\psi(x) = 2 \sum_{n \le x} \left\{ \sum_{k \le x/n} \mathrm{L}1^{*1/2}(k) \right\} \mu^{*1/2}(n) \,. \tag{2}$$

Below, we shall apply Condition (1) to show that

$$2 \sum_{k \le x/n} \mathrm{L}1^{*1/2}(k) \approx \sum_{k \le x/n} \{1 * 1^{*1/2} + c' 1^{*1/2} * 1^{*1/2} + c'' 1^{*1/2}\}(k). \tag{3}$$

Insert this into (2), use $1^{*1/2} * \mu^{*1/2} = \delta$, and we get

$$\begin{aligned} \psi(x) &= \sum_{n \le x} (\{1 * 1^{*1/2} + c' 1^{*1/2} * 1^{*1/2} + c'' 1^{*1/2}\} * \mu^{*1/2})(n) + o(x) \\ &= \sum_{n \le x} \{1(n) + c' \, 1^{*1/2}(n) + c'' \delta(n)\} + o(x) \sim x. \end{aligned}$$

2 d. Discussion

The preceding method works if we have estimate (1) for $\sum 1^{*1/2}$ and can give a suitable error analysis for $\sum \mathrm{L}1^{*1/2}$ in (3).

Why does this approach fail if we attempt to treat

$$\psi(x) = \sum_{n \le x} \Big\{ \sum_{m \le x/n} \mathrm{L}1(m) \Big\} \mu(n)$$

by approximating $\sum \mathrm{L}1$ by $\sum (1 * 1 + c1)$?
The problem here is that *a priori* we know only $\sum_{n \le x} \mu(n) \ll x$, while for $\mu^{*1/2}$ we have the better bound

$$\Big| \sum_{n \le x} \mu^{*1/2}(n) \Big| \le \sum_{n \le x} 1^{*1/2}(n) \ll x \log^{-1/2} x.$$

2 e. Estimate of formula (3)

Assuming

$$\sum_{n\le x} 1^{*1/2}(n) = \frac{x}{\log^{1/2} x} + \frac{c\,x}{\log^{3/2} x} + o\left(\frac{x}{\log^{3/2} x}\right), \qquad (1\text{ bis})$$

summing by parts yields

$$\sum_{n\le x} \mathrm{L}1^{*1/2}(n) = x\log^{1/2} x + \frac{c' x}{\log^{1/2} x} + o\left(\frac{x}{\log^{1/2} x}\right).$$

In place of $\sum_{n\le x}(1^{*1/2} * 1)(n)$, use the (equivalent) expression

$$I := \int_1^x dN_{1/2} * dt = \int_1^x N_{1/2}(x/t)\,dt.$$

Using (1 bis) and easy estimates we find

$$I = 2cx\log^{1/2} x + c'' x + \frac{c''' x}{\log^{1/2} x} + o\left(\frac{x}{\log^{1/2} x}\right).$$

2 f. Estimate of formula (3) continued

Also, trivally,

$$\sum_{n \le x} (1^{*1/2} * 1^{*1/2})(n) = \sum_{n \le x} 1(n) = \lfloor x \rfloor,$$

and

$$\sum_{n \le x} 1^{*1/2}(n) = \frac{x}{\log^{1/2} x} + o\Big(\frac{x}{\log^{1/2} x}\Big) .$$

Thus, with suitable choices of c_2, c_3, we have

$$\begin{aligned} \Delta(x) &:= 2 \sum_{n \le x} \mathrm{L}1^{*1/2}(n) - \int_1^x dN_{1/2} * dt - c_2 \lfloor x \rfloor - c_3 N_{1/2}(x) \\ &= o\Big(\frac{x}{\log^{1/2} x}\Big) . \end{aligned}$$

2 g. Conclusion of the proof of the PNT

From the formula for ψ, the estimate for Δ, and the relation $1^{*1/2} * \mu^{*1/2} = \delta$ we find

$$\psi(x) = \sum_{n \le x} (2\,\mathrm{L}1^{*1/2} * \mu^{*1/2})(n) = 2 \sum_{n \le x} \sum_{m \le x/n} \mathrm{L}1^{*1/2}(m)\,\mu^{*1/2}(n)$$

$$= \int_1^x dt \;+\; c_2 N_{1/2}(x) + c_3 + E(x) \;=\; x + o(x) + E(x),$$

where

$$E(x) := \sum_1^x \Delta(x/n)\mu^{*1/2}(n) \ll \sum_1^x o\Big(\frac{x/n}{\log^{1/2} x/n}\Big)\, 1^{*1/2}(n).$$

Using the bound for $N_{1/2}$ again and making a simple estimate, we find $E(x) = o(x)$, and thus $\psi(x) \sim x$.

3 a. Estimate of $\sum 1^{*1/2}$

Recall Dirchlet divisor function estimate

$$\sum_{n \leq x} 1 * 1(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

We establish

$$N_{1/2}(x) := \sum_{n \leq x} 1^{*1/2}(n) \sim \frac{x}{\log^{1/2} x} + \frac{cx}{\log^{3/2} x} \qquad \text{(1 bis)}$$

using Perron inversion formula, assuming a modest zero-free region for $\zeta(s)$, the Riemann zeta function. For non-integral x,

$$N_{1/2}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} x^s \zeta(s)^{1/2} ds/s \,, \qquad (4)$$

with $\mathcal{C}$ a vertical line to the right of 1 in $\mathbb{C}$.

3 b. Estimate of $\sum 1^{*1/2}$ continued

The main contribution to the integral arises from the half-order pole of $\zeta(s)^{1/2}$ at $s = 1$. We evaluate the integral by deforming the contour $\mathcal{C}$ to extend a bit to the left of the line $\{\Re s = 1\}$ but which encircles $s = 1$, and then using the Hankel loop integral formula.

We find for non-integral x,

$$N_{1/2}(x) \sim x \log^{-1/2} x + (\gamma/2 - 1) x \log^{-3/2} x.$$

Note that we used non-vanishing of $\zeta(s)$ and Mellin inversion, so, despite how it began, *this proof of the PNT is not elementary.*

4 a. Suitability of our hypothesis

Our proof of the PNT assumed that we know the first two terms of the asymptotic series for $N_{1/2}(x)$. Here we indicate that this is an appropriate condition.

Suppose the PNT were false. Then, by familiar theory, $\zeta(s)$ would have zeros at some points $s = 1 \pm i\alpha$. Again use Perron's formula

$$N_{1/2}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} x^s \zeta(s)^{1/2} ds/s \qquad \text{(4 bis)}$$

and take account of the half-order zeros. At $1 + i\alpha$, the Hankel loop integral formula gives an additional term $ke^{i\theta} x^{1+i\alpha} \log^{-3/2} x$. This and the conjugate together give a contribution

$$2kx\, cos(\alpha \log x + \theta) \log^{-3/2} x,$$

i.e. there would be a cosine wobble in the $x \log^{-3/2} x$ term for $N_{1/2}(x)$, so condition (1) could not hold.

4 b. A tauberian ingredient in our proof

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Usual proofs of PNT contain a tauberian element. Here is ours:

The formula

$$\sum_{n \le x} 1^{*1/2} * 1^{*1/2}(n) = \lfloor x \rfloor$$

is an entangling of two copies of $1^{*1/2}$. Passage from $1^{*1/2} * 1^{*1/2}$ to the asymptotic estimate

$$\sum_{n \le x} 1^{*1/2}(n) \sim x \log^{-1/2} x + cx \log^{-3/2} x \qquad \text{(1 bis)}$$

is a tauberian activity.