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Some Results on Metric Trees

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Metric Trees

Let M be a metric space with metric d.

Definition 1. For $x, y \in M$, a *geodesic segment* from x to y is the image of an isometric embedding $\alpha : [a, b] \to M$ such that $\alpha(a) = x$ and $\alpha(b) = y$. The geodesic segment will be called *metric segment* and is denoted by $[x, y]$.

Definition 2. We call M a *metric tree* if for any $x, y, z \in M$, the following hold:

1. there is a unique metric segment between x and y.
2. if $[x, z] \cap [z, y] = \{z\}$, then $[x, z] \cup [z, y] = [x, y]$.

The following is an example of metric tree.

Example 1. Let ρ denote the Euclidean metric on $\mathbb{R}^2$. We define the *radial metric* by

$$d(x,y) = \begin{cases} \rho(x,y) & \text{if } y = tx \text{ for some } t \in \mathbb{R} \\ \rho(x,0) + \rho(y,0) & \text{otherwise.} \end{cases}$$

Definition 3. Let M be a metric tree, and let $A \subseteq M$. We call $F_A = \{f \in A : f \notin (x, y)$ for all $x, y \in A\}$ the set of *final points* of A. Here, $(x, y) = [x, y] \backslash \{x, y\}$.

This concept leads us to a characterization of compact metric trees.

Theorem 1. *(Aksoy, Borman, Westfahl) A metric tree (M, d) is compact if and only if*

$$M = \bigcup_{f \in F_M} [a, f] \text{ for all } a \in M, \text{ and } \overline{F}_M \text{ is compact.}$$

Hyperconvexity

Definition 4. A metric space M is called *hyperconvex* if $\bigcap_{i \in I} B_c(x_i, r_i) \neq \emptyset$ for any collection $\{B_c(x_i; r_i)\}_{i \in I}$ of closed balls in X with $x_i x_j \leq r_i + r_j$.

Theorem 2. *(Aronszajn and Panitchpakdi) X is a hyperconvex metric space if and only if for all metric spaces D, if $C \subset D$ and $f : C \to X$ is a nonexpansive mapping, then f can be extended to the nonexpansive mapping $\tilde{f} : D \to X$.*

The simplest example of hyperconvex space is the set of real numbers $\mathbb{R}$ or a finite dimensional real Banach space endowed with the maximum norm. While the Hilbert space ℓ^2 fails to be hyperconvex, the spaces ℓ^∞ and L^∞ are hyperconvex. The connection between hyperconvex metric spaces and metric trees is given in the following theorem:

Theorem 3. *(Aksoy and Kirk) A complete metric tree T is hyperconvex.*

Embeddings of Metric Trees

Definition 5. Let X be a metric space. Define $\ell^\infty(X)$, as the normed linear space where

$$\ell^\infty(X) := \left\{ (x_m)_{m \in X} \mid x_m \in \mathbb{R}, \sup_{m \in X} |x_m| < \infty \right\}$$

and $\|(x_m)\| := \sup_{m \in X} |x_m|$.

Theorem 4. *(Kirk and Sims) Let X be a metric space and $a \in X$, then*

$$J : X \to \ell^\infty(X) \text{ where } J(x) = (xm - am)_{m \in X}$$

is an isometric embedding of X into $\ell^\infty(X)$.

We define a "canonical" embedding $J = J_{x^*}$ of tree T into $\ell_\infty(T)$ (x^* is a point in T) by

$$J_{x^*}(x)(y) = J(x)(y) = d(x, y) - d(x^*, y).$$

When T is finite, we can also use the embedding

$$J(x)(y) = d(x, y).$$

We also use the "semicanonical" embedding of T into ℓ_1 (for such embedding see Section 2.5 of S. Evans, *Probability and real trees*). We have not been able to construct explicit embeddings of metric trees into other Banach spaces. However, we can show:

Theorem 5. *Suppose X is a superreflexive Banach space, T is a finitely generated metric tree, and $\varepsilon > 0$. Then there exists a Banach space Y, $(1 + \varepsilon)$-isomorphic to X, such that T embeds into Y isometrically.*

Barycenters in Metric Trees

Suppose U is an isometric embedding of a metric tree T into a Banach space X, equipped with the norm $\|\cdot\|$. Suppose $x_1, \ldots, x_n \in T$, and let $\tilde{x}_0 = (x_1 + \ldots + x_n)/n$ be their barycenter in X (we identify $x \in T$ with $U(x) \in X$). Let $\mathbf{P} = \mathbf{P}_{U,T,X}$ be the set of contractive retractions π from X onto $U(T)$ (it is non-empty since T is injective). We try to describe $\mathbf{P}(\tilde{x}_0)$. More generally, suppose $\alpha = (\alpha_i)_{i=1}^n$ is a sequence of positive numbers, with $\sum_{k=1}^n \alpha_k = 1$. Set $\tilde{x}^{(\alpha)} = \sum_{k=1}^n \alpha_k x_k$, and try to describe $\mathbf{P}(\tilde{x}^{(\alpha)})$.

Theorem 6. *Suppose T is a complete metric tree, embedded into $\ell_\infty(T)$ in the canonical way. For $x_0 \in T$, the following are equivalent:*

1. *$x_0 \in \mathbf{P}(\tilde{x}^{(\alpha)})$.*
2. *$\|x_0 - x\| \leq \sum_k \alpha_k \|x_k - x\|$ for any $x \in T$.*

Theorem 7. *Suppose T is a finitely generated tree, embedded into ℓ_1 in the semicanonical way. For $x_0 \in T$, the following are equivalent:*

1. *$x_0 \in \mathbf{P}(\tilde{x}^{(\alpha)})$.*
2. *$\|x_0 - x\| \leq \sum_k \alpha_k \|x_k - x\|$ for any $x \in T$.*

Proposition 8. *Suppose a complete metric tree T is embedded isometrically into a normed space X, and $\tilde{x}$ is a point of X. Then $\mathbf{P}(\tilde{x})$ is a closed, metric convex subset of T.*

Type and Cotype

We say that a metric space (X, d) satisfies *the four-point inequality* if, for any $x_1, x_2, x_3, x_4 \in X$,

$d(x_1, x_2) + d(x_3, x_4) \leq$
$\max\{d(x_1, x_3) + d(x_2, x_4), d(x_1, x_4) + d(x_2, x_3)\}$.

Theorem 9. *For type and cotype of metric trees we have:*

1. *Any metric space satisfying the four-point inequality has metric type 2, with constant 1. In particular, this result holds for metric trees.*
2. *Any complete metric tree has metric cotype 2, with a universal constant.*

Entropy Quantities

Definition 6. In the following, for ϵ-cover sets of diameter $\leq 2\epsilon$ and for ϵ-net balls of radius ϵ is used.

1. Let $\mathcal{N}_\epsilon(A)$ be the cardinality of a minimal ϵ-cover of A. Then define the *ϵ-entropy of A* as
$$\mathcal{H}_\epsilon(A) := \log_2 \mathcal{N}_\epsilon(A).$$
Similarly, let $\mathcal{N}_\epsilon^M(A)$ be the cardinality of a minimal ϵ-net for A in M. Then define the *ϵ-entropy of A relative to M* as
$$\mathcal{H}_\epsilon^M(A) := \log_2 \mathcal{N}_\epsilon^M(A).$$
2. A is *centered* if for all $U \subset A$ such that $\operatorname{diam}(U) = 2r$, there exists $a \in A$ such that $U \subset B_c(a; r)$.
3. Given a normed linear space X and subset A, define the *nth Kolmogorov diameter (n-width) of A in X* as:
$$\delta_n(A) := \inf \left\{ \sup_{a \in A} d(a, M_n) \mid M_n \text{ is a } n\text{-dim subsp. of } X \right\}.$$
4. Let T be a metric tree and $A \subset T$, then define the *nth Kolmogorov diameter of A in $\ell^\infty(T)$* as:
$$\delta_n(A) := \delta_n(J(A), \ell^\infty(T)).$$

Recall that for a metric tree T, we have the isometric embedding $J : T \to \ell^\infty(T)$. If T is a complete metric tree, then T is hyperconvex and thus injective so we have a nonexpansive projection $P : \ell^\infty(T) \to T$, and where $P \circ J = \mathrm{id}_T$.

Theorem 10. *For entropy quantities and other measures of noncompactness in metric trees we have:*

1. *Every complete metric tree T is centered.*
2. *For a complete metric tree T, if $A \subset T$, then*
$$\mathcal{H}_\epsilon^T(A) = \mathcal{H}_\epsilon(A).$$
3. *Suppose S is a compact subset of a complete metric tree T, and $\varepsilon_1, \varepsilon_2$ are positive numbers. Then*
$$\mathcal{N}_{\varepsilon_1+\varepsilon_2}(con(S)) \leq \mathcal{N}_{\varepsilon_1}(S) \lceil \operatorname{diam} S/(4\varepsilon_2) \rceil.$$
4. *If T is a complete metric tree, where $A \subset T$ bounded, then*
$$\lim_{n \to \infty} \delta_n(A) = \beta(A).$$
Where $\beta(A)$ is the Hausdorff measure of noncompactness defined as
$$\beta(A) := \inf \left\{ b > 0 \mid A \subset \bigcup_{j=1}^n B(x_j; b) \text{ for some } x_j \in T \right\}.$$

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