1. Let $T$ be an arbitrary metric space and $f : \mathbb{R} \times T \to \mathbb{R}$ a function. Assume that $f(.,t)$ is a measurable function for each $t \in T$ and $f(x,.)$ continuous function for each $x \in \mathbb{R}$. Assume also that there exists an integrable function $g$ such that for each $t \in T$ we have $|f(x,t)| \leq g(x)$ for almost all $x \in \mathbb{R}$. Show that the function $F : T \to \mathbb{R}$ defined by

$$F(t) = \int_{\mathbb{R}} f(x,t) \, dx$$

is a continuous function.

2. Let $f$ be integrable over $(-\infty, \infty)$, then show that

a) $\int f(x)dx = \int f(x+t)dx$

b) Let $g$ be bounded measurable function. Then show that

$$\lim_{t \to 0} \int_{-\infty}^{\infty} |g(x)[f(x) - f(x+t)]| = 0$$

3.

a) Let $\{f_n\}$ be a sequence of real valued measurable functions. If $\{f_n\}$ converges to $f$ in measure , show that $\{f_n\}$ is a Cauchy sequence in measure.

b) Show that if a sequence $\{f_n\}$ of integrable functions converge to $f$ in $L^1$, then $\{f_n\}$ converges $f$ in measure. Is the converse true?

Note: We say $\{f_n\}$ is **Cauchy in measure** if for every $\epsilon > 0$,

$$m(\{x : |f_n(x) - f_m(x)| > \epsilon\}) \to 0 \quad \text{as} \quad m,n \to \infty$$

and we say $\{f_n\}$ is **converges in measure** to a measurable function $f$ if for every $\epsilon > 0$

$$m(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0 \quad \text{as} \quad n \to \infty$$
4. Compute the following limits and justify the calculations:

a) \( \lim_{n \to \infty} \int_0^{\infty} \left[ 1 + \left( \frac{x}{n} \right) \right]^{-n} \sin\left( \frac{x}{n} \right) dx \)

b) \( \lim_{n \to \infty} \int_0^{\infty} \frac{n \sin\left( \frac{x}{n} \right)}{x(1 + x^2)} dx \)

c) \( \lim_{n \to \infty} \int_a^{\infty} \frac{n(1 + n^2 x^2)}{x} dx \)

5. Show that \( \int_0^{\infty} x^{2n} e^{-x^2} dx = \frac{(2n)!}{2^{2n} n!} \cdot \frac{\sqrt{\pi}}{2} \) holds true for \( n = 0, 1, 2, \ldots \)

Hint: Use induction on \( n \) and the fact that \( \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \) (Euler's formula)

6. Show that for \( a > 0 \), \( \int_{-\infty}^{\infty} e^{-x^2} \cos(ax) dx = \sqrt{\pi} e^{-a^2/4} \)

Hint: use problem 5 above.

7. Suppose that \( f \) is real continuously differentiable function on \([a, b]\), \( f(a) = f(b) = 0 \) and that \( \int_a^b f^2(x) dx = 1 \). Prove that:

a) \( \int_a^b x f(x) f'(x) dx = -1/2 \)

b) \( \int_a^b \left[ f'(x) \right]^2 \cdot \int_a^b x^2 f^2(x) dx \geq 1/4 \)

Hint for part b): recall that \( \langle f, g \rangle = \int_a^b f(x) g(x) dx \) defines an inner product on \( C[a, b] \).