1. Given a subset $E$ of $\mathbb{R}$, define exterior measure of $E$ by

$$m_*(E) = \inf \{ \sum_{n=1}^{\infty} l(I_n) : E \subset \bigcup_{n=1}^{\infty} I_n \}$$

where the infimum is taken over all coverings of $E$ by countable union of bounded intervals. Show the following:

(a) $0 \leq m_*(E) \leq \infty$

(b) If $E \subset F$, then $m_*(E) \leq m_*(F)$

(c) $m_*(E + x) = m_*(E)$ where $E + x = \{ e + x : e \in E \}$

(d) $m_*(E) = 0$ for any countable set $E$.

(e) $m_*(E) < \infty$ for any bounded set $E$.

(f) $m_*(E) = \inf \{ \sum_{n=1}^{\infty} (b_n - a_n) : E \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \}$

2. Show that $m_*(I) = \ell(I)$ for any interval $I$ bounded or not.

3. For an arbitrary subset $E$ of $\mathbb{R}$ and real numbers $a, b$ define the set

$$aE + b = \{ ax + b : x \in A \}$$

a) $m_*(aE + b) = |a|m_*(E)$

b) If $E$ is measurable, then so is $aE + b$.

4. Let $E = \mathbb{Q} \cap [0, 1]$ Let $\{ I_n \}$ be a finite collection of open intervals covering $E$. Show that

$$\sum_{i=1}^{\infty} |I_i| \geq 1.$$ What can you conclude from this question?

5. Let $E$ be a subset of $\mathbb{R}$ with $m_*(E) > 0$. Show that there exists a nonmeasurable set $F$ of $\mathbb{R}$ such that $F \subseteq E$. 

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6. Give an example of pairwise disjoint sequence \( \{E_n\} \) of measurable sets such that

\[
m_*(\bigcup_{n=1}^{\infty} E_n) < \sum_{n=1}^{\infty} m_*(E_n).
\]

7. Consider the family of sets \( \{E_n\}_{n=1}^{\infty} \). We define \textbf{limit superior} and \textbf{limit inferior} as follows:

\[
\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n
\]

\[
\liminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{n=k}^{\infty} E_n
\]

a) Show that

\[
\limsup E_n = \{x : x \in E_n \text{ for infinitely many } n\}
\]

and

\[
\liminf E_n = \{x : x \in E_n \text{ for all but finitely many } n\}.
\]

b) Prove the \textbf{Borel-Cantelli lemma}: If \( \{E_n\}_{n=1}^{\infty} \) is a countable family of measurable subsets of \( \mathbb{R}^d \) such that \( \sum_{n=1}^{\infty} m(E_n) < \infty \) then \( E = \limsup E_n \) is measurable and \( m(E) = 0 \).

Note: In probability theory the Borel-Cantelli lemma is stated as follows: Given \( \{E_n\} \) sequence of events in some probability space, if the sum of the probabilities of \( E_n \) is finite (i.e., \( \sum_{n=1}^{\infty} m(E_n) < \infty \)), then the probability that infinitely many of them occur is 0. (i.e., \( m(\limsup E_n) = 0 \)).