Math 137/331 - Real Analysis I
HW 7

1. Let F be a closed subset in $\mathbb{R}$, the distance from x to F, $d(x, F)$ is defined as:

$$f(x) = d(x, F) = \inf\{|x - y| : \; y \in Y\}$$

Prove that $\dfrac{f(x+y)}{|y|} \to 0$ for almost a.e. $x \in F$

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2. Suppose F is of bounded variation and continuous. Prove that $F = F_1 - F_2$, where both F_1 and F_2 are monotonic and continuous.

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3. One-sided Hardy Littlewood maximal function f_+^* is defined as

$$f_+^*(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)| dy$$

Show that $m(E_\alpha^+) = \dfrac{1}{\alpha} \displaystyle\int_{E_\alpha^+} |f(y)| dy$, where $E_\alpha^+ = \{x \in \mathbb{R} : \; f_+^* > \alpha\}$.
Hint: Consider $F(x) = \int_0^x |f(y)| dy - \alpha x$, apply rising sun lemma (lemma 3.5) to this function , to see $E_\alpha^+ = \displaystyle\bigcup_{j=1}^{\infty} (a_j, b_j)$ and $F(a_j) = F(b_j)$.

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4. Show that if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then

(a) f maps sets of measure zero to sets of measure zero.

(b) f maps measurable sets to measurable sets.

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5. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq M|x - y|$$

for some M and all $x, y \in \mathbb{R}$, if and only if f satisfies the following properties:

(a) f is absolutely continuous.

(b) $|f'(x)| \leq M$ for a.e. x.

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6. If $a, b > 0$, let $f(x) = x^a \sin(x^{-b})$ for $0 < x \leq 1$ and $f(0) = 0$. Prove that f is of bounded variation in $[0, 1]$ if and only if $a > b$.

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7. Show that the set of discontinuities of a monotone function is at-most countable.

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8. Let $f : [a, b] \to \mathbb{R}$ be differentiable function. If the derivative f' is uniformly bounded on $[a, b]$, then show that f' is Lebesgue integrable and that

$$\int_{[a,b]} f' dx = f(b) - f(a).$$

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