Math 137/331 Real Analysis I
HW 5

1. Let T be an arbitrary metric space and $f : \mathbb{R} \times T \to \mathbb{R}$ a function. Assume that $f(., t)$ is a measurable function for each $t \in T$ and $f(x, .)$ continuous function for each $x \in \mathbb{R}$. Assume also that there exists an integrable function g such that for each $t \in T$ we have $|f(x,t)| \leq g(x)$ for almost all $x \in \mathbb{R}$. Show that the function $F : T \to \mathbb{R}$ defined by

$$F(t) = \int_{\mathbb{R}} f(x,t) \ dx$$

is a continuous function.

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2. Let f be integrable over $(-\infty, \infty)$, then show that

a)

$$\int f(x)dx = \int f(x+t)dx$$

b) Let g be bounded measurable function. Then show that

$$\lim_{t \to 0} \int_{-\infty}^{\infty} |g(x)[f(x) - f(x+t)]| = 0$$

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3.

a) Let $\{f_n\}$ be a sequence of real valued measurable functions. If $\{f_n\}$ converges to f in measure , show that $\{f_n\}$ is a Cauchy sequence in measure.

b) Show that if a sequence $\{f_n\}$ of integrable functions converge to f in L^1, then $\{f_n\}$ converges f in measure. Is the converse true?

Note: We say $\{f_n\}$ is **Cauchy in measure** if for every $\epsilon > 0$,

$$m(\{x : \ |f_n(x) - f_m(x)| > \epsilon\}) \to 0 \quad \text{as } m, n \to \infty$$

and we say $\{f_n\}$ is **converges in measure** to a measurable function f if for every $\epsilon > 0$

$$m(\{x : \ |f_n(x) - f(x)| > \epsilon\}) \to 0 \quad \text{as } n \to \infty$$

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4. Compute the following limits and justify the calculations:

a) $\lim_{n\to\infty} \int_0^\infty [1 + (x/n)]^{-n} \sin(x/n) dx$

b) $\lim_{n\to\infty} \int_0^\infty n \sin(x/n)[x(1 + x^2)]^{-1} dx$

c) $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1} dx$

5. Show that $\int_0^\infty x^{2n} e^{-x^2} dx = \frac{(2n)!}{2^{2n} n!} \cdot \frac{\sqrt{\pi}}{2}$ holds true for $n = 0, 1, 2, \cdots$

Hint: Use induction on n and the fact that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (Euler's formula)

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6. Show that for $a > 0$, $\int_{-\infty}^\infty e^{-x^2} \cos(ax)\, dx = \sqrt{\pi} e^{-a^2/4}$

Hint: use problem 5 above.

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7. Suppose that f is real continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$ and that $\int_a^b f^2(x) dx = 1$. Prove that:

a) $\int_a^b x f(x) f'(x) dx = -1/2$

b) $\int_a^b [f'(x)]^2 \cdot \int_a^b x^2 f^2(x) dx \geq 1/4$

Hint for part b): recall that $< f, g >= \int_a^b f(x) g(x)\, dx$ defines an inner product on $C[a, b]$.

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