Math 137/331 Real Analysis I
HW 4

1. Prove Tchebychev inequality. Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : \ f(x) > \alpha \}$, prove that

$$m(E_\alpha) \leqq \frac{1}{\alpha} \int f$$

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2.

a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is integrable, but f^2 is not.

b) Show that a measurable function f is integrable if and only if $|f|$ is integrable. Give an example of a nonintegrable function whose absolute value is integrable.

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3. Suppose f is Riemann integrable on the closed interval $[a, b]$. Then show that f is measurable and

$$(R) \int_{[a,b]} f = (L) \int_{[a,b]} f$$

where the integral on the left-hand side is the Riemann Integral, and that on the right-hand side is the Lebesgue integral.

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4. Let μ be counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorems as statements about infinite series.

5. If f_n, g_n, f, g are all integrable functions (all in L^1) with $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$, then show that $\int f_n \to \int f$
Hint: Rework the proof of the dominated convergence theorem.

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6. Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then show that $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.
Hint: use Problem 5 above.

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7. Suppose $f \geq 0$, let

$$\mu(E) = \int_E f\, dm$$

for a measurable set E.

1. Show that μ is a measure.
2. Show that for any $g \geq 0$

$$\int g\, d\mu = \int f g\, dm$$

   Hint: First suppose g is simple.

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8. Show that $f(x) = \dfrac{\ln x}{x^2}$ is Lebesgue integrable over $[1, \infty)$ and that $\int f dx = 1$.

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9. Show that the improper Riemann integral

$$\int_0^\infty \cos(x^2) dx$$

exists but not Lebesgue integrable over $[0, \infty)$.

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10. Establish the Riemann-Lebesgue Theorem: If f is integrable function on $(-\infty, \infty)$ then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos(nx) dx = 0$$

Hint: Theorem is easy if f is a simple function, then use theorem 2.4-Stein, page 71.

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