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Math Analysis II
HW 2
Due 02/07/2013

1. Let $f_n(x) = \frac{nx}{1+nx}$ for $x \geq 0$

   a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

   b) Show that for $a > 0$ convergence is uniform on $[a, \infty]$.

   c) Show that convergence is not uniform on $[0, \infty]$.

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2. Let $f_n = \frac{1}{1+x^n}$ for $x \in [0, 1]$

   a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

   b) Show that if $0 < a < 1$, the convergence is uniform on $[0, a]$.

   c) Show that the convergence is not uniform on $[0, 1]$.

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3. Determine whether the sequence $\{f_n\}$ converges uniformly on D.

   a) $f_n(x) = \frac{1}{1+(nx-1)^2}$ $\quad D = [0, 1]$

   b) $f_n(x) = nx^n(1-x)$ $\quad D = [0, 1]$

   c) $f_n(x) = \arctan \frac{2x}{x^2+n^3}$ $\quad D = \mathbb{R}$

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4. Suppose that the sequence $\{f_n\}$ converges uniformly to f on the set D and that for each $n \in \mathbb{N}$, f_n is bounded on D. Prove that f is bounded on D

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5. Suppose a sequence of functions $\{f_n\}$ are defined as

$$f_n(x) = 2x + \frac{x}{n} \qquad x \in [0, 1]$$

   a) Find the limit function f.

   b) Is f continuous on [0,1]?

   c) Does $[\lim f_n(x)]' = \lim f_n'(x)$ for $x \in [0, 1]$?

   d) Does $\int_0^1 \lim f_n(x)dx = \lim \int_0^1 f_n(x)dx$?

6. Give examples to illustrate that

a) there exists differentiable functions f_n and f such that $f_n \to f$ pointwise on [0,1] but

$$\lim_{n\to\infty} f_n'(x) \neq \left(\lim_{n\to\infty} f_n(x)\right)' \quad x = 1$$

b) there exist continuous functions f_n and f such that $f_n \to f$ pointwise on [0,1] but

$$\lim_{n\to\infty} \int_0^1 f_n(x) \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)$$

7. Discus the uniform convergence of the following series

a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $\mathbb{R}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{\sqrt{n}}$ on $[0, 2\pi]$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(nx)}{n^2}$ on $\mathbb{R}$

8. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence r, where $0 < r \leq +\infty$. If $0 < t < r$, prove that the power series converges uniformly on $[-t, t]$.

9. Suppose a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has a uniformly continuous derivative on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} n[f(x + \frac{1}{n}) - f(x)] = f'(x)$$

10. Let $f_n : [1,2] \longrightarrow \mathbb{R}$ be defined by

$$f_n(x) = \frac{x}{(1+x)^n}$$

a) Show that $\sum_{n=1}^{\infty} f_n(x)$ converges for $x \in [0,2]$.

b) Use Dini's Theorem to show that the convergence is uniform.

c) Does the following hold:

$$\int_1^2 \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x) dx$$

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