Math 138/332 Real Analysis II
HW 6 - Due April 19, 2011

1. Let X be a normed linear space and B be a Banach space. Let M be a dense subspace of X and F_0 bounded linear map from M into B. Show that there exists a unique bounded linear map $F : X \to B$ such that

$$F_M = F_0 \quad \text{and} \quad ||F|| = ||F_0||.$$

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2. Show that if a normed linear space X is reflexive then it is reflexive in any equivalent norm.

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3. Let $E \subseteq C[0,1]$ be a closed linear subspace consisting of only C^1 functions. Prove that E is finite dimensional.
Hint: Consider the differentiation operator $D : E \to C[0,1]$ defined by $D(f) = f'$. Use closed graph theorem to show D is continuous. Then show B_E is uniformly bounded. Refer to Arzela-Ascoli Theorem to assure that $B_E \subseteq (C[0,1], ||.||_\infty)$ is relatively compact. Riesz Theorem implies E is finite dimensional.

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4. On $C[0,1]$ consider the following norms:

$$||f||_\infty = \sup\{|f(x)| : \ x \in [0,1]\} \quad \text{and} \quad ||f||_1 = \int_0^1 |f(x)|\, dx$$

Show that the identity operator $I : (C[0,1], ||f||_\infty) \to (C[0,1], ||f||_1)$ is continuous, onto but not open. Why does this not contradict the open mapping theorem?

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5. Let (T_n) be a sequence of bounded linear maps of a Banach space X into another Banach space Y. Assume that for each $y^* \in Y^*$ and $x \in X$ there is a constant k such that $|y^*(T_n x)| \leq k$ for each $n \in \mathbb{N}$. If there is a dense subset A of X for which $(T_n x)$ converges for every $x \in A$, show that $(T_n x)$ converges for every $x \in X$.
Hint: It is enough to show $(T_n x)$ is Cauchy in Y. Use uniform boundedness principle.

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6. Let X and Y be Banach spaces and $T : X \to Y$ is a linear map.For all $x \in X$ define

$$||x||_1 = ||x|| + ||Tx||.$$

Let

a) T has closed graph

b) $(X, ||.||_1)$ is a Banach space

c) T is bounded.

Show that $a) \Leftrightarrow b)$ and $a) \Leftrightarrow c)$.

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7. Let X be a vector space of all real valued functions on $[0, 1]$ having continuous first order derivatives. Show that $||f|| = |f(0)| + ||f'||_\infty$ is a norm on X that is equivalent to the norm $||f||_\infty + ||f'||_\infty$.
Hint: Use $f(x) = f(0) + \displaystyle\int_0^x f'(t)\ dt$.

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8.

a) State Baire's Theorem

b) Show that the set of irrational numbers is not a countable union of closed subsets of $\mathbb{R}$.
Hint: First observe closed and proper vector subspace of a normed space is nowhere dense. Then assume the contrary to part b) and use Baire's thm since $\mathbb{R}$ is a complete metric space.

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