1. Prove that the conjugate space of \( c_0 \) is \( \ell^1 \). That is \( c_0^* = \ell^1 \) where
\[
c_0 = \{ x = (x_n) : x_n \to 0 \text{ as } n \to \infty \}.
\]

2. 
   a) Let \( X \) be a linear space and \( Y \subseteq X \) a linear subspace. Prove that each linear functional \( f : Y \to \mathbb{K} \) has a linear extension \( \hat{f} : X \to \mathbb{K} \).
   
   b) Let \( X \) be a normed space, \( n \in \mathbb{N} \) and \( \{x_1, \ldots, x_n\} \subseteq X \) a linearly independent system. Prove that for any \( \alpha_1, \ldots, \alpha_n \in \mathbb{K} \) there is \( x^* \in X^* \) such that \( x^*(x_i) = \alpha_i \) for all \( 1 \leq i \leq n \).

3. Show that if \( X \) and \( Y \) are nontrivial normed spaces and \( \mathcal{B}(X,Y) \) is a Banach space, then \( Y \) is a Banach space.
   Hint: Let \( y_n \) be a Cauchy sequence in \( Y \). Pick \( f \in X^* \), consider the sequence of operators \( \{T_n\} \) of \( \mathcal{B}(X,Y) \) define \( T_n(x) = f(x)y_n \).

4. Show that a linear functional \( f \) on a normed space \( X \) is discontinuous if and only if for each \( a \in X \) and each \( r > 0 \), we have
\[
f(B(a; r)) = \{ f(x) : ||a - x|| < r \} = \mathbb{K}
\]
   Hint: Note that \( B(a; r) = a + rB(0; 1) \). Recall that \( f \) is continuous if and only if \( f \) is bounded.

5. Let \( M \) be a closed subspace of a normed linear space \( X \) and let \( x_0 \) be a vector not in \( M \). If \( d \) is the distance from \( x_0 \) to \( M \), then show that there exists a functional \( f_0 \) in \( X^* \) such that
\[
f_0(M) = 0, \quad f_0(x_0) = 1 \quad \text{and} \quad ||f_0|| = \frac{1}{d}
\]
   Hint: Define \( f \) on \( M_0 = M + [x_0] \) by \( f(y) = f(x + \alpha x_0) = \alpha \)
6. A subset $A$ in a normed space is called **total** if the smallest subspace containing $A$ is dense in $X$. Prove that $A$ is total if and only if for $f \in X^*$ and $f(a) = 0$ for each $a \in A$ implies that $f = 0$. 

7. Let $H$ be a Hilbert space, and $G \subseteq H$ a closed linear subspace. Prove that any linear and continuous functional on $G$ has a unique Hahn-Banach extension on $H$.

Hint: If $f : G \to \mathbb{K}$ a linear and continuous functional, then show that $\tilde{f} : H \to \mathbb{K}$ defined by $\tilde{f}(x) = f(P_G(x))$ is the unique Hahn-Banach of $f$. Here $f(P_G(x))$ is the orthogonal projection of $x$ onto $G$. For uniqueness consider a bounded linear functional $g : H \to \mathbb{K}$ where $g$ restricted to $G$ is $f$ and $\|f\| = \|g\|$. Apply Riesz Representation theorem to $g$ and show $\tilde{f} = g$. 
