Math 138/332 Real Analysis II
HW 5 - Due April 5, 2011

1. Prove that the conjugate space of $c_0$ is $\ell^1$. That is $c_0^* = \ell^1$ where

$$c_0 = \{x = (x_n) : \ x_n \to 0 \text{ as } n \to \infty \}.$$

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2.

a) Let $X$ be a linear space and $Y \subseteq X$ a linear subspace. Prove that each linear functional $f : Y \to \mathbb{K}$ has a linear extension $\tilde{f} : X \to \mathbb{K}$.

b) Let $X$ be a normed space, $n \in \mathbb{N}$ and $\{x_1, \ldots x_n\} \subseteq X$ a linearly independent system. Prove that for any $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ there is $x^* \in X^*$ such that $x^*(x_i) = \alpha_i$ for all $1 \le i \le n$.

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3. Show that if $X$ and $Y$ are nontrivial normed spaces and $\mathbf{B}(X, Y)$ is a Banach space, then $Y$ is a Banach space.
Hint:Let $y_n$ be a Cauchy sequence in $Y$. Pick $f \in X^*$, consider the sequence of operators $\{T_n\}$ of $\mathbf{B}(X, Y)$ define $T_n(x) = f(x)y_n$.

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4. Show that a linear functional $f$ on a normed space $X$ is discontinuous if and only if for each $a \in X$ and each $r > 0$, we have

$$f(B(a; r)) = \{f(x) : \ ||a - x|| < r\} = \mathbb{R}$$

Hint:Note that $B(a; r) = a + rB(0; 1)$. Recall that $f$ is continuous if and only if $f$ is bounded.

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5. Let $M$ be a closed subspace of a normed linear space $X$ and let $x_0$ be a vector not in $M$. If $d$ is the distance from $x_0$ to $M$, then show that there exists a functional $f_0$ in $X^*$ such that

$$f_0(M) = 0, \ \ f_0(x_0) = 1 \text{ and } ||f_0|| = \frac{1}{d}$$

Hint: Define $f$ on $M_0 = M + [x_0]$ by $f(y) = f(x + \alpha x_0) = \alpha$

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6. A subset $A$ in a normed space is called **total** if the smallest subspace containing $A$ is dense in $X$. Prove that $A$ is total if and only if for $f \in X^*$ and $f(a) = 0$ for each $a \in A$ implies that $f = 0$.

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7. Let $H$ be a Hilbert space, and $G \subseteq H$ a closed linear subspace. Prove that any linear and continuous functional on $G$ has a unique Hahn-Banach extension on $H$.
Hint: If $f : G \to \mathbb{K}$ a linear and continuous functional, then show that $\widetilde{f} : H \to \mathbb{K}$ defined by $\widetilde{f}(x) = f(P_G(x))$ is the unique Hahn-Banach of $f$. Here $f(P_G(x))$ is the orthogonal projection of $x$ onto $G$. for uniqueness consider a bounded linear functional $g : H \to \mathbb{K}$ where $g$ restricted to $G$ is $f$ and $||f|| = ||g||$. Apply Riesz Representation theorem to $g$ and show $\widetilde{f} = g$.

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