1. 
   a) If $X$ is a linear space on which two norms which generate the same topology on $X$, prove that either both of them complete, or none of them complete.
   
   b) Define $d_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $d_1(x, y) = |x - y|$, $d_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $d_2(x, y) = |\phi(x) - \phi(y)|$, where $\phi(x) = \frac{x}{(1 + |x|)}$, $\forall x \in \mathbb{R}$. Prove that $d_1$ and $d_2$ are distances on $\mathbb{R}$, which generate the same topology, but $(\mathbb{R}, d_1)$ is complete and $(\mathbb{R}, d_2)$ is not complete.

2. Let $X$ be a normed space, $Y \subseteq X$ be a closed linear subspace such that $Y$ (with the norm from $X$) is complete and $X/Y$ (with the quotient norm) is complete. Prove that $X$ is complete.

   Recall that the quotient space $X/Y$ is a normed space with respect to quotient norm $||x|| = \inf\{||y|| : x - y \in Y\}$ where $x = x + Y$.

3. Let $X$ be a normed space. Prove that $X$ is a Banach space if and only if any decreasing sequence of closed balls from $X$ with the sequence of radii converging to 0 has non-empty intersection.

   Note that this is Cantor type of characterization of a Banach space.

4. Prove that a Banach space having a Schauder basis is separable.

   Hint: If a normed space $X$ contains a sequence $(e_n)$ with the property that for every $x \in X$, there is a unique sequence of scalars $(a_n)$ such that
   
   $$||x - (a_1 e_1 + a_2 e_2 + \ldots + a_n e_n)|| \to 0 \quad \text{as} \quad n \to \infty,$$

   then $(e_n)$ is called a Schauder basis for $X$. The series $\sum_{k=1}^{\infty} a_k e_k$ which is the sum $x$ is called the expansion of $x$ with respect to $(e_n)$ and we write $x = \sum_{k=1}^{\infty} a_k e_k$. 

   ■
5. Let $X$ and $Y$ be normed linear spaces and let $T$ be a linear transformation of $X$ into $Y$. Prove the following statements are equivalent.

a) $T$ is bounded.
b) $T$ is uniformly continuous.
c) $T$ is continuous at some point of $X$.

6. Prove that the set of all continuously differentiable functions $C^1[0,1]$ defined on $[0,1]$ is a Banach space under the norm

$$||f|| = ||f||_\infty + ||f'||_\infty$$

for $f \in C^1[0,1]$ where $||f||_\infty = \sup |f(x)|$ and $||f'||_\infty = \sup |f'(x)|$ for $x \in [0,1]$.

7. Let $BV[a,b]$ denote the set of all complex or real valued functions of bounded variations on $[a,b]$. For $f \in BV[a,b]$ define

$$||f|| = f(a) + V(f)$$

where $V$ is the total variation of $f$ defined as $V(f) = \sup \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$ for different partitions of $[a,b]$. Show that the above defined $||f||$ is actually a norm on $BV[a,b]$, then show that $(BV[a,b], ||f||)$ is a Banach space.

8. Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ so that any bounded linear transformation on $\mathcal{B}(X, Y)$ is represented by an $m \times n$ matrix $A = (a_{ij})$. Then prove the following:

a) If $X$ and $Y$ are endowed with the uniform norm $||.||_u$ then $||T|| = \max_{1 \leq i \leq m} \left( \sum_{j=1}^n |a_{ij}| \right)$
b) $X$ and $Y$ are endowed with the 1-norm $||.||_1$ then $||T|| = \max_{1 \leq j \leq n} \left( \sum_{i=1}^m |a_{ij}| \right)$.

9. Show that the adjoint operator $T^* H \to H$ where $H$ is a Hilbert space has the following properties:

$$a) (T_1 + T_2)^* = T_1^* + T_2^*, \quad b) (aT)^* = \overline{a} T^*, \quad c) (T_1 T_2)^* = T_1^* T_2^*,$$

$$d) T^{**} = T^*, \quad e) ||T^*|| = ||T||, \quad f) ||TT^*|| = ||T||^2.$$