Math 138/332 Real Analysis II
HW 4 - Due March 22, 2011

1.

a) If X is a linear space on which two norms which generate the same topology on X, prove that either both of them complete, or non of them complete.

b) Define $d_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $d_1(x, y) = |x - y|$, $d_2\mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $d_2(x, y) = |\phi(x) - \phi(y)|$, where $\phi(x) = \dfrac{x}{(1 + |x|)}$, $\forall x \in \mathbb{R}$. Prove that d_1 and d_2 are distances on $\mathbb{R}$, which generate the same topology, but $(\mathbb{R}, d_1)$ is complete and $(\mathbb{R}, d_2)$ is not complete.

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2. Let X be a normed space, $Y \subseteq X$ be a closed linear subspace such that Y (with the norm from X) is complete and X/Y (with the quotient norm) is complete. Prove that X is complete.

Recall that the quotient space X/Y is a normed space with respect to quotient norm $||\overline{x}|| = \inf\big\{||y|| =: \ x - y \in Y\big\}$ where $\overline{x} = x + Y$.

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3. Let X be a normed space. Prove that X is a Banach space if and only if any decreasing sequence of closed balls from X with the sequence of radii converging to 0 has non-empty intersection.
Note that this is **Cantor type** of characterization of a Banach space.

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4. Prove that a Banach space having a Schauder basis is separable.
Hint: If a normed space X contains a sequence (e_n) with the property that for every $x \in X$, there is a unique sequence of scalars (a_n) such that

$$||x - (a_1e_1 + a_2e_2 + \ldots + a_ne_n)|| \to 0 \quad \text{as} \quad n \to \infty,$$

then (e_n) is called a **Schauder basis** for X. The series $\displaystyle\sum_{k=1}^{\infty} a_k e_k$ which is the sum x is called the expansion of x with respect to (e_n) and we write $x = \displaystyle\sum_{k=1}^{\infty} a_k e_k$.

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5. Let X and Y be normed linear spaces and let T be a linear transformation of X into Y. Prove the following statements are equivalent.

a) T is bounded.

b) T is uniformly continuous.

c) T is continuous at some point of X.

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6. Prove that the set of all continuously differentiable functions $C'[0,1]$ defined on $[0,1]$ is a Banach space under the norm

$$||f|| = ||f||_\infty + ||f'||_\infty \text{ for } \ f \in C'[0,1]$$

where $||f||_\infty = \sup|f(x)|$ and $||f'||_\infty = \sup|f'(x)|$ for $x \in [0,1]$.

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7. Let $BV[a,b]$ denote the set of all complex or real valued functions of bounded variations on $[a,b]$. for $f \in BV[a,b]$ define

$$||f|| = f(a) + V(f)$$

where V is the total variation of f defined as $V(f) = \sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1}|$ for different partitions of $[a,b]$. Show that the above defined $||f||$ is actually a norm on $BV[a,b]$, then show that $(BV[a,b], ||f||)$ is a Banach space.

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8. Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ so that any bounded linear transformation on $\mathfrak{B}(X,Y)$ is represented by an $m \times n$ matrix $A = (a_{ij})$. Then prove the following:

a) If X and Y are endowed with the uniform norm $||.||_u$ then $||T|| = \max_{1 \leq i \leq m}\big(\sum_{j=1}^{n} |a_{ij}|\big)$

b) X and Y are endowed with the 1-norm $||.||_1$ then $||T|| = \max_{1 \leq j \leq n}\big(\sum_{j=1}^{m} |a_{ij}|\big)$.

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9. Show that the adjoint operator $T^*H \to H$ where H is a Hilbert space has the following properties:

$$a)\,(T_1 + T_2)^* = T_1 * + T_2^*, \ \ b)\,(aT)^* = \overline{a}T^*, \ \ c)\,(T_1T_2)^* = T_1^*T_2^*,$$

$$d)\,T^{**} = T^*, \ \ e)\,||T^*|| = ||T||, \ \ f)\,||TT^*|| = ||T||^2.$$