1. Let \( X \neq 0 \) be a real or complex linear space. Prove that there is at least one norm on \( X \).

Hint. Every linear space has a basis.

2. Given a function \( p : X \to [0, \infty) \) with the properties
   
   a) \( p(x) = 0 \iff x = 0 \)
   
   b) \( p(\lambda x) = |\lambda|p(x) \) for all \( x \in X \) and \( \lambda \in K \).

Show that \( p \) is a norm if and only if \( B_X = \{x \in X : p(x) \leq 1\} \) is convex.

(i.e., The triangle axiom and the convexity of the closed unit ball are equivalent)

3. Let \( a > 0 \). On \( C[0,1] \) consider the following norms:

\[
||f||_\infty = \sup_{t \in [0,1]} |f(t)|
\]

\[
||f||_1 = a \int_0^1 |f(t)| dt.
\]

Prove that \( ||f|| = \min\{||f||_\infty, ||f||_1\} \) is a norm on \( C[0,1] \) if and only if \( a \leq 1 \).

This problem shows minimum of two norms is not a norm in general. Show that maximum of two norms is indeed a norm.

4. Let \( X = M_{n,m}(\mathbb{R}) \) be the real vector space of \( n \times m \) matrices with real entries. Given \( A, B \in M_{n,m}(\mathbb{R}) \), set

\[
(A, B) = tr(A'B)
\]

where by “\( tr \)” we mean the trace of a square matrix. i.e., the sum of the entries lying in the diagonal.

   a) Show that \((.,.)\) is an inner product on \( M_{n,m}(\mathbb{R}) \).

   b) Deduce that \( A \mapsto ||A|| = \sqrt{tr(A'A)} \) is a norm on \( X \).
5. Show that every Hilbert space is uniformly convex.
A normed linear space is said to be uniformly convex if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ independent of $x$ and $y$ such that $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x - y\| \geq \epsilon$ implies $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$.

6. 
   a) Suppose $f$ is supported on a set $E$ of finite measure. If $f \in L^2(\mathbb{R}^n)$, then show that $f \in L^1(\mathbb{R}^n)$ and that
   
   $\|f\|_{L^1(\mathbb{R}^n)} \leq m(E)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^n)}$

   b) If $|f(x)| \leq M$ ( $f$ is bounded) and $f \in L^1(\mathbb{R}^n)$, then $f \in L^2(\mathbb{R}^n)$ and that

   $\|f\|_{L^2(\mathbb{R}^n)} \leq M^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{2}}$

7. Show that $L^2(\mathbb{R}^n)$ is separable. i.e., There exists a countable dense set in $L^2(\mathbb{R}^n)$. 