Your Name
Math Analysis II
HW 9
May 2, 2013

1. Investigate whether the system:
Let $0 < a < b$ and

$$f(x) = \begin{cases} 1 & \text{if } x \in [a,b] \cap \mathbb{Q} \\ 0 & \text{if } x \in [a,b] \text{ is irrational.} \end{cases}$$

Find the upper and lower Riemann integrals of $f(x)$ over $[a,b]$.

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2. Show that a monotonic function f on $[a,b]$ is Riemann integrable on $[a,b]$

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3. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Let $g : [a,b] \to \mathbb{R}$ be a function such that $\{x \in [a,b]; f(x) \neq g(x)\}$ is finite. Show that $g(x)$ is Riemann integrable and

$$\int_a^b f(x)dx = \int_a^b g(x)dx \ .$$

Does the conclusion still hold when $\{x \in [a,b]; f(x) \neq g(x)\}$ is countable?

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4. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Show that $|f(x)|$ is Riemann integrable and

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx \ .$$

When do we have equality?

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5.Use Riemann sums to find the following limits

1. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^4}{n^5}$

2. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{9n^2 + k^2}$

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6. Let $f : [0,1] \to \mathbb{R}$ be C^2, i.e. $f(x)$ is twice differentiable and $f''(x)$ is continuous. Find

$$\lim_{n\to\infty} n^2 \int_0^1 f(x)dx - n\sum_{k=1}^{n} f\left(\frac{2k-1}{2n}\right) .$$

Hint: From Taylor's formula we know that

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(\theta)}{2}(y-x)^2 ,$$

for any $x, y \in [0,1]$ for some θ between x and y. Also use $\displaystyle\int_0^1 f(x)dx = \sum_{k=1}^{k=n} \int_{(k-1)/n}^{k/n} f(x)dx.$

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7. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Show that there exists $c \in (a,b)$ such that

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c) .$$

Is this still true for Riemann integrable functions?

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8. et $f : [a,b] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_a^b f(x)\sin(nx)dx = 0 \text{ and } \lim_{n\to\infty} \int_a^b f(x)\cos(nx)dx = 0.$$

Use these limits to find

$$\lim_{n\to\infty} \int_a^b f(x)\sin^2(nx)dx .$$

This is known as Riemann-Lebesgue's lemma.

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9.

a) Show that if a set A has volume zero than A has measure zero.

b) Show that if A is compact and measure zero, then measure of the boundary of A is also zero, $\displaystyle\int_A \chi_A$ exists and volume of A is also zero.

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10. Let $f_k \to f$ **uniformly** on $A \subset \mathbb{R}^n$. Show that

a) {discontinuities of f}$\subset \bigcup_{k=1}^{\infty}$ {discontinuities of f_k }

b) Prove that if f_k is a sequence of bounded Riemann integrable functions on A such that $f_k \to f$ **uniformly** on A, then f is Riemann integrable.

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