Your Name
Math Analysis II
HW 6
03/26/13

Let f be a function on $[a, b]$ that is differentiable at x_0.Let $T(x)$ be the tangent line to f at x_0. Then prove that T is the unique linear function with the property that

$$\lim_{x \to x_0} \frac{f(x) - T(x)}{x - x_0} = 0$$

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Assume f and g are differentiable at a. Find:

1. $\lim_{x \to a} \frac{xf(a) - af(x)}{x - a}$
2. $\lim_{x \to a} \frac{f(x)g(a) - f(a)g(x)}{x - a}$

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Let f be differentiable at a. Find:

1. $\lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x - a}$ where $n \in N$.
2. $\lim_{n \to \infty} n(f(a + \frac{1}{n}) + f(a + \frac{2}{n} + ... + f(a + \frac{k}{n} - kf(a))$ where $k \in N$.

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If f is differentiable on an interval $[a, b]$ and α satisfies

$$f'(a) < \alpha < f'(b)$$

(or $f'(a) > \alpha > f'(b)$) then show that there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.
Hint: Define $g(x) = f(x) - \alpha x$ and show that $g'(c) = 0$ for some $c \in (a, b)$.

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Use the Mean Value Theorem for functions of one variable to prove the following inequalities

1. $sinx < x$ and $cosx > 1 - \frac{x^2}{2}$ for $0 < x \leq \frac{\pi}{2}$
2. $x - \frac{x^3}{6} < sinx$ and $cosx < 1 - \frac{x^2}{2} + \frac{x^4}{24}$ for $0 < x \leq \frac{\pi}{2}$.

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1. Let f be a C'' function on $[0, 1]$ with $f(0) = f(1) = 0$, and suppose that $|f''(x)| \leq A$ for all x, $0 < x < 1$. Show that
$$|f'(\frac{1}{2}| \leq \frac{A}{4}$$
and that
$$|f'(x)| \leq \frac{A}{2}$$
for $0 < x \leq 1$.

2. If $f(0) = 0$ and $|f'(x)| \leq M|f(x)|$ for $0 \leq x \leq L$, show that on that interval $f(x) \equiv 0$.

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Find the second order Taylor polynomial for the following functions at the indicated points.

1. $f(x, y) = \frac{1}{x^2+y^2+1}$ at $\overrightarrow{a} = (0, 0)$.
2. $f(x, y) = e^{2x+y}$ at $\overrightarrow{a} = (0, 0)$.
3. $f(x, y) = e^{2x} cos 3y$ at $\overrightarrow{a} = (0, \pi)$.

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Find the Hessian matrix $Hf(\overrightarrow{a})$ for the following functions at the indicated points.

1. $f(x, y) = \frac{1}{x^2+y^2+1}$ at $\overrightarrow{a} = (0, 0)$.
2. $f(x, y, z) = x^3 + x^2y - yz^2 + 2z^3$ at $\overrightarrow{a} = (1, 0, 1)$.

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Find the third-order Taylor polynomial $P_3(x, y, z)$ for $f(x, y, z) = e^{x+2y+3z}$ at $(0, 0, 0)$.

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A function $f : R \to R$ is called <u>analytic</u> function provided that
$$f(x + h) = f(x) + f'(x)h + ... + \frac{f^k(x)}{k!}h^k + ...$$
i.e, the series on the right-hand side converges and equals to $f(x + h)$.
Show that the function defined as:
$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } \ x > 0 \\ 0 & \text{otherwise} \end{cases}$$
is a C^∞ function , but f is not analytic.

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