Your Name
Math Analysis II
HW 4
02/21/13

1.

a) Let $f : [1, 2] \to [0, 3]$ be a continuous function with$f(1) = 0$ and $f(2) = 3$. Show that f has a fixed point.

b) Let $f : [a, b] \to [a, b]$ be a Lipschitz function with Lipschitz constant $0 < L < 1$. Show that f has a unique fixed point.

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2.

a) Give an example of a mapping T of a complete metric space into itself with the property $d(Tx, Ty) < d(x, y)$ for all x, y with $x \neq y$ which has no fixed point.

b) Show that $f : R \to R$ defined as $f(x) = cosx$ is not a contraction but that the function$g(x) = \frac{99}{100} cosx$ is a contraction.

c) Show that *cosine* function is a contraction mapping on $[0, a]$ for any $1 \leq a < \frac{\pi}{2}$. Using the Banach Contraction Mapping Theorem find the solution to the equation $x = cosx$ correct to three decimal places.

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3. Let $X = \{x \in Q : x \geq 1\}$ and $f : X \to X$ is defined by $f(x) = \frac{x}{2} + \frac{1}{x}$.

a) Show that for all $x, y \in X$ one has $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$. Where d is the usual metric on real numbers.

b) Show that for this contraction f there is no $x \in X$ for which $x = f(x)$. Does this contradict BCMT ?

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4. Show that the system of equations:
$x_1 = \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{2}{15}x_3 + 3$
$x_2 = \frac{1}{4}x_1 + \frac{1}{5}x_2 + \frac{1}{2}x_3 - 1$
$x_3 = -\frac{1}{4}x_1 + \frac{1}{3}x_2 - \frac{1}{3}x_3 + 2$
has a unique solution.

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5. Let $T : C[0,1] \to C[0,1]$ be a mapping defined by

$$Tf(x) = \int_0^x f(s)ds.$$

Show that:

a) T is NOT a contraction.

b) T has a unique fixed point.

c) T^2 is a contraction.
Here $C[0,1]$ will denote the normed space of continuous functions with uniform norm.

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6.

a) Let B_c be a closed ball in a complete metric space M, and let $T : B_c \to M$ be a contraction which moves the center of B_c a distance at most $(1-k)r$ where r is the radius of B_c and k is the contraction constant. Show that T has a unique fixed point and it is in B_c.

b) Let B be an open ball in a complete metric space M, and let $T : B \to M$ be a contraction which moves the center of B a distance less than $(1-k)r$ where r is the radius of B and k is the contraction constant. Show that T has a unique fixed point.

c) Let T be a contraction on a complete metric space M, and suppose that T moves the point x a distance d. Show that the distance from x to the fixed point is at most $\frac{d}{(1-k)}$, where k is the contraction constant.

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7. Consider R^n with a metric $d(x,z) = \sum_{i=1}^{n} |x_i - z_i|$. Let T be a mapping from R^n to itself defined by the system of linear equations

$$y_i = \sum_{j=1}^{n} a_{ij}x_j + b_i$$

where $i = 1, 2, ...n$. Under what conditions is T a contraction? What will be the condition if the above metric is replaced by the Euclidean metric $d(x,z) = [\sum_{i=1}^{n} |x_i - z_i|^2]^{\frac{1}{2}}$

8. If T is a mapping from a complete metric space (M,d) into itself such that T^m is a contraction mapping for some $m \in N$, then show that T has a unique fixed point.

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9.

a) Convert the initial value problem (IVP) to an integral equation and set up an iteration scheme to solve it.

$$\frac{dy}{dx} = 3xy \quad \text{where} \quad y(0) = 1$$

b) Given the initial value problem

$$f'(x) = 1 + x - f(x) \quad \text{for} \quad \frac{-1}{2} \le x \le \frac{1}{2} \quad \text{where } f(0) = 1$$

First show the mapping $T : C[\frac{-1}{2}, \frac{1}{2}] \to C[\frac{-1}{2}, \frac{1}{2}]$ defined by

$$Tf(x) = 1 + x + \frac{1}{2}x^2 - \int_0^x f(t)dt$$

is a contraction, then set up an iteration scheme to solve it.

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10. Let f be a real valued twice continously differentiable function on $[a, b]$. Let $\widetilde{x}$ be a simple zero of f in (a,b). Show that Newton's method defined by:

$$x_{n+1} = g(x_n) \quad \text{and} \quad g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

is a contraction in some neighborhood of $\widetilde{x}$, so that the iterative sequence converges to $\widetilde{x}$ for any x_0 sufficiently close to $\widetilde{x}$.
Note that $\widetilde{x}$ is a simple zero implies that $f'(x) \neq 0$ on some neighborhood U of $\widetilde{x}$ where $U \subset [a, b]$ and f'' is bounded on U.

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