Your Name
Math Analysis II
HW 3
February 14, 2013

Prove the following

a) Prove that if a sequence $\{f_n\}$ of continuous functions on A converges uniformly on A, then f is continuous on A.

b) Prove that $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is continuous on $[0, 1]$.

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Study the convergence and uniform convergence of $\{f_n\}$ and $\{\acute{f_n}\}$ on A where

a) $f_n = \frac{sin(nx)}{\sqrt{n}}$ where $A = \mathbb{R}$

b) $f_n = \frac{x}{1+n^2x^2}$ where $A = [-1, 1]$

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Let $\{a_n\}$ be a sequence of real numbers, and let $\{f_n\}$ be a sequence of functions satisfying

$$\sup\{|f_n(x) - f_m(x)| : \ x \in A\} \leq |a_n - a_m|$$

where $n, m \in N$. Prove that $\{f_n\}$ converges uniformly on A.

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Assume that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$, and there is $M > 0$ such that $|f(x)| < M$ and $|g(x)| < M$ for all $x \in A$. Show that $f_n g_n \rightrightarrows fg$.

Hint: Consider $|f_n g_n - fg + fg_n - fg_n|$.

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Use Drichlet test (given below) to show that the following series converges uniformly on the indicated set.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ where $A=[0,1]$.

2. $\sum_{n=1}^{\infty}\frac{sin(nx)}{n}$ where $A=[\delta,2\pi-\delta],\ 0<\delta<\pi$.

[Drichlet Test for convergence: Assume that $f_n, g_n : A \to \mathbb{R},\ n \in N$ satisfy the following conditions:

1. For each fixed $x \in A$, the sequence $\{f_n(x)\}$ is monotonic

2. $\{f_n(x)\}$ converges uniformly to zero on A

3. The sequence of partial sums of $\sum_{n=1}^{\infty} g_n(x)$ is uniformly bounded on A.

Then the series $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly on A]

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Let g be a continuous function satisfying the Lipschitz condition in the second variable, and let $T : \mathcal{C}[a,b] \to \mathbb{R}$ be defined by

$$Tx = \int_a^b g(t, y(t))dt$$

for $x \in \mathcal{C}[a,b]$.

a) Prove that T is continuous.

b) Show that if we restrict T to the compact subsets of $\mathcal{C}[a,b]$, then there exist a function x such that $\int_a^b g(t,x(t))dt$ is a minimum.

Note that g be a continuous function satisfying the Lipschitz condition in the second variable means, there is a constant $M>0$ such that

$$|g(t,x(t)) - g(t,y(t))| \le M|x(t)-y(t)|$$

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For $f : A \to \mathbb{R}$, with $A \subset \mathbb{R}$ define

$$w_f(\delta) = \sup\{|f(x_1) - f(x_2)| : \ x_1, x_2 \in A, \ |x_1 - x_2| < \delta\}.$$

We call w_f the modulus of continuity of f and observe that w_f is monotonically increasing on $(0, \infty)$ and thus

$$\lim_{\delta \to 0^+} w_f(\delta) = \inf_{\delta > 0} w_f(\delta) \geq 0.$$

Show that f is uniformly continuous if and only if $\lim_{\delta \to 0^+} w_f(\delta) = 0$

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Consider the following sets of sequences of functions. Explain why Arzela-Ascoli Theorem may fail.

1. $\mathbb{B} = \{f_n : \ f_n(x) = x + n \text{ where } x \in [0,1]\}$
2. $\mathbb{B} = \{f_n : \ f_n(x) = x^n \text{ where } x \in [0,1]\}$
3. $\mathbb{B} = \{f_n : \ f_n(x) = \frac{1}{1+(x-n)^2} \text{ where } x \in [0,\infty)\}$

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Assume that $f_n : [a,b] \to \mathbb{R}$ is a sequence of differentiable functions whose derivatives are uniformly bounded. If for some x_0, $f_n(x_0)$ is bounded as $n \to \infty$ then show that the sequence $\{f_n\}$ has a subsequence that converges uniformly on $[a,b]$.

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