Prof. Asuman Aksoy
Math Analysis I
HW 3
Due 02/9/2012

1)

a) Show that the set of irrational numbers in $(0, 1)$ is not countable.

b) Show that $\mathbb{R}$ is uncountable

c) Show that any nonempty subset of a countable set is finite or countable.

■

2)

a) An algebraic number is a root of a polynomial, whose coefficients are rational. Show that the set of all algebraic numbers is countable.

Hint: Use the Fundamental Theorem of Algebra: A polynomial of degree n can have at most n roots. You may also need the fact that countable union of finite sets is countable.

b) Prove that the collection of transcendental numbers is uncountable (two famous transcendental numbers are π and e).
Hint: Any number is either algebraic or transcendental.

■

3)

a) Show that if $A_1, A_1, \cdots, A_n$ are countable, then $A_1 \times A_1 \times \cdots \times A_n$ is countable.

b) What can you say about the countable Cartesian product of countable sets?

■

4) Given any set A show that there does **not** exist a function $f : A \to \mathcal{P}(A)$ that is onto.

Hint: Prove by contradiction. Assume $f : A \to \mathcal{P}(A)$ is onto. Notice that f is a correspondence between a set and its power set. Therefore the assumption that f is onto means that every subset of A appears as $f(a)$ for some $a \in A$. To arrive at a contradiction, produce a subset $B \subseteq A$ that is not equal to $f(a)$ for any $a \in A$.

■

5)
Give a strategy for choosing N in terms of ϵ to show that:

a) $\lim_{n\to\infty} \frac{1}{n^2} = 0$

b) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

■

6)

a) Let (x_n) and (a_n) be sequences of real numbers and let $x \in \mathbb{R}$. Suppose for some $k > 0$ and some $m \in \mathbb{N}$, we have

$$|x_n - x| \leq k|a_n| \quad \text{for all } n > m,$$

and $\lim_{n\to\infty} a_n = 0$. Show that $\lim_{n\to\infty} x_n = x$

b) Show that if $\{x_n\}$ converges to l, then $\{|x_n|\}$ converges to $|l|$. What about the converse?

■

7) Suppose (x_n) is a sequence in $\mathbb{R}$ defined by

$$x_n = \int_1^n \frac{cost}{t^2} dt$$

a) Show that $|x_n| \leq 1$ for $n = 1, 2, 3, \cdots$

b) Show that the sequence (x_n) is Cauchy.

Hint: You can use $|\int_1^n \frac{cost}{t^2} dt| \leq \int_1^n |\frac{cost}{t^2}| dt$ and indefinite integral of $\frac{1}{t^2}$ is $\frac{1}{t}$.

■

8) Let $\{x_n\}$ be a sequence such that there exist $A > 0$ and $C \in (0, 1)$ for which

$$|x_{n+1} - x_n| \leq AC^n$$

for any $n \geq 1$. Show that $\{x_n\}$ is Cauchy. Is this conclusion still valid if we assume only

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0$$

Hint: Choose $m = n + k$ and show that $|x_n - x_{n+k}| < \frac{A}{1-C} C^n$, recalling the following fact about geometric series :

$$a + ar + ar^2 + \cdots + ar^n = a.\frac{1 - r^{n+1}}{1 - r} < \frac{a}{1 - r} \quad \text{if} \quad 0 < r < 1.$$

■

9) Show that $\{x_n\}$ defined by

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is divergent.

Hint: Show that $\{x_n\}$ fails to be Cauchy by showing that $\frac{1}{2} \leq x_{2n} - x_n$.

■