1. 
   a) Let $f : [1, 2] \to [0, 3]$ be a continuous function with $f(1) = 0$ and $f(2) = 3$. Show that $f$ has a fixed point.
   
   b) Let $f : [a, b] \to [a, b]$ be a Lipschitz function with Lipschitz constant $0 < L < 1$. Show that $f$ has a unique fixed point.

2. 
   a) Give an example of a mapping $T$ of a complete metric space into itself with the property $d(Tx, Ty) < d(x, y)$ for all $x, y$ with $x \neq y$ which has no fixed point.
   
   b) Show that $f : R \to R$ defined as $f(x) = \cos x$ is not a contraction but that the function $g(x) = \frac{99}{100} \cos x$ is a contraction.
   
   c) Show that cosine function is a contraction mapping on $[0, a]$ for any $1 \leq a < \frac{\pi}{2}$. Using the Banach Contraction Mapping Theorem find the solution to the equation $x = \cos x$ correct to three decimal places.

3. Let $X = \{x \in Q : x \geq 1\}$ and $f : X \to X$ is defined by $f(x) = \frac{x}{2} + \frac{1}{x}$.
   
   a) Show that for all $x, y \in X$ one has $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$. Where $d$ is the usual metric on real numbers.
   
   b) Show that for this contraction $f$ there is no $x \in X$ for which $x = f(x)$. Does this contradict BCMT?

4. 
   Show that the system of equations:
   
   $x_1 = \frac{1}{4} x_1 - \frac{1}{2} x_2 + \frac{2}{5} x_3 + 3$
   $x_2 = -\frac{1}{4} x_1 + \frac{1}{5} x_2 + \frac{2}{3} x_3 - 1$
   $x_3 = -\frac{1}{4} x_1 + \frac{1}{3} x_2 - \frac{1}{3} x_3 + 2$

   has a unique solution.
5. Let \( T : C[0,1] \to C[0,1] \) be a mapping defined by
\[
Tf(x) = \int_0^x f(s)ds.
\]
Show that:

a) \( T \) is NOT a contraction.

b) \( T \) has a unique fixed point.

c) \( T^2 \) is a contraction.

Here \( C[0,1] \) will denote the normed space of continuous functions with uniform norm.

6. 

a) Let \( B_c \) be a closed ball in a complete metric space \( M \), and let \( T : B_c \to M \) be a contraction which moves the center of \( B_c \) a distance at most \((1 - k)r\) where \( r \) is the radius of \( B_c \) and \( k \) is the contraction constant. Show that \( T \) has a unique fixed point and it is in \( B_c \).

b) Let \( B \) be an open ball in a complete metric space \( M \), and let \( T : B \to M \) be a contraction which moves the center of \( B \) a distance less than \((1 - k)r\) where \( r \) is the radius of \( B \) and \( k \) is the contraction constant. Show that \( T \) has a unique fixed point.

c) Let \( T \) be a contraction on a complete metric space \( M \), and suppose that \( T \) moves the point \( x \) a distance \( d \). Show that the distance from \( x \) to the fixed point is at most \( d \frac{1}{(1-k)} \), where \( k \) is the contraction constant.

7. 
Consider \( R^n \) with a metric \( d(x, z) = \sum_{i=1}^{n} |x_i - z_i| \). Let \( T \) be a mapping from \( R^n \) to itself defined by the system of linear equations
\[
y_i = \sum_{j=1}^{n} a_{ij}x_j + b_i
\]
where \( i = 1, 2, \ldots, n \). Under what conditions is \( T \) a contraction? What will be the condition if the above metric is replaced by the Euclidean metric \( d(x, z) = \left[ \sum_{i=1}^{n} |x_i - z_i|^2 \right]^{1/2} \)

8. 
If \( T \) is a mapping from a complete metric space \((M, d)\) into itself such that \( T^m \) is a contraction mapping for some \( m \in N \), then show that \( T \) has a unique fixed point.
9. Convert the initial value problem (IVP) to an integral equation and set up an iteration scheme to solve it.

\[ \frac{dy}{dx} = 3xy \quad \text{where} \quad y(0) = 1 \]

10. Let \( f \) be a real valued twice continuously differentiable function on \([a, b]\). Let \( \tilde{x} \) be a simple zero of \( f \) in \((a, b)\). Show that Newton’s method defined by:

\[ x_{n+1} = g(x_n) \quad \text{and} \quad g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \]

is a contraction in some neighborhood of \( \tilde{x} \), so that the iterative sequence converges to \( \tilde{x} \) for any \( x_0 \) sufficiently close to \( \tilde{x} \).

Note that \( \tilde{x} \) is a simple zero implies that \( f'(x) \neq 0 \) on some neighborhood \( U \) of \( \tilde{x} \) where \( U \subset [a, b] \) and \( f'' \) is bounded on \( U \).