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Math Analysis I
HW 7
Due 04/04/2013

1. Discuss whether following sets are open or closed. Determine the interiors, closures and boundaries of each sets.

a) $(1, 2)$ in $\mathbb{R}$.

b) $[2, 3]$ in $\mathbb{R}$.

c) $\bigcap_{n=1}^{\infty}[-1, \frac{1}{n})$ in $\mathbb{R}$.

d) $(0, 1) \cap \mathbb{Q}$ in $\mathbb{R}$.

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2. Let $A$ be a subset of a metric space $(M, d)$. Define

$$\breve{A} := \{a \in A : a \text{ is an interior point of } A\} \text{ and } int(A) := \bigcup\{O \subseteq A : O \text{ is open in } M\}.$$

Show that

a) $\breve{A} = int(A)$

b) $A$ is open if and only if $A = int(A)$.

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3. Let $A$ be a subset of $\mathbb{R}^n$. Show that

a) $int(A) \subset A \subset \overline{A}$

b) $A$ is closed if and only if $A = \overline{A}$.

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4.Let $A \subset \mathbb{R}$ be nonempty and bounded above. Let $x = \sup(A)$. Show that $x \in \overline{A}$ and $x \in \partial A$.

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5. For a subset $A$ of a metric space, show that $x \in \overline{A}$ if and only if

$$d(x, A) = \inf\{d(x, y) : y \in A\} = 0.$$

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6. Prove that an element $x \in (M, d)$ is an accumulation point of $A$ if and only if there is a sequence $\{x_n\} \subset A \backslash \{x\}$ such that $\lim_{n\to\infty} x_n = x$.

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7.Prove that a sequence $(x_n)$ in a normed space is a Cauchy sequence if and only if for every neighborhood $U$ of 0 there is an $N$ such that $m, n > N$ implies that $x_n - x_m \in U$.

Note that A **neighborhood** of a point in a normed space is an open set containing that point.

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8. Identify $\mathbb{R}^{n+m}$ with $\mathbb{R}^n \times \mathbb{R}^m$. Show that $A \subset \mathbb{R}^{n+m}$ is open if and only if for each $(x, y) \in A$ with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, there exists open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ with $x \in U$, $y \in V$ such that $U \times V \subset A$. Deduce that product of open sets is open.

Hint: Consider $(x, y) \in A$ and the disk $D((x, y); \epsilon)$. Define $D_x$ to be the projection of $D((x, y); \epsilon)$ onto its first $n$ components and $D_y$ be the projection onto the last $m$ components.

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