Prof. Asuman Aksoy
Math Analysis I
HW 3
Due 02/14/2013

1. Prove the following Inequalities

a) $|x+y| \leq |x| + |y|$ (Triangle Inequality)

b) $||x| - |y|| \leq |x-y|$ (Alternate Triangle Inequality)

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2.

a) Let r be a rational and t be an irrational number. Prove that rt is irrational.

b) Given any two real numbers x and y with $x < y$, show that there exists an irrational number t satisfying $x < t < y$.

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3. Give a strategy for choosing N in terms of ϵ to show that:

a) $\lim_{n\to\infty} \frac{1}{n^2} = 0$

b) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

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4.

a) Let (x_n) and (a_n) be sequences of real numbers and let $x \in \mathbb{R}$. Suppose for some $k > 0$ and some $m \in \mathbb{N}$, we have

$$|x_n - x| \leq k|a_n| \text{ for all } n > m,$$

and $\lim_{n\to\infty} a_n = 0$. Show that $\lim_{n\to\infty} x_n = x$

b) Show that if $\{x_n\}$ converges to l, then $\{|x_n|\}$ converges to $|l|$. What about the converse?

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5. Suppose (x_n) is a sequence in $\mathbb{R}$ defined by

$$x_n = \int_1^n \frac{cost}{t^2} dt$$

a) Show that $|x_n| \leq 1$ for $n = 1, 2, 3, \cdots$

b) Show that the sequence (x_n) is Cauchy.

Hint: You can use $|\int_1^n \frac{cost}{t^2} dt| \leq \int_1^n |\frac{cost}{t^2}| dt$ and indefinite integral of $\frac{1}{t^2}$ is $\frac{1}{t}$.

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6. Let $\{x_n\}$ be a sequence such that there exist $A > 0$ and $C \in (0, 1)$ for which

$$|x_{n+1} - x_n| \leq AC^n$$

for any $n \geq 1$. Show that $\{x_n\}$ is Cauchy. Is this conclusion still valid if we assume only

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0$$

Hint: Choose $m = n + k$ and show that $|x_n - x_{n+k}| < \frac{A}{1-C} C^n$, recalling the following fact about geometric series :

$$a + ar + ar^2 + \cdots + ar^n = a.\frac{1 - r^{n+1}}{1 - r} < \frac{a}{1 - r} \quad \text{if} \quad 0 < r < 1.$$

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7. Show that $\{x_n\}$ defined by

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is divergent.

Hint: Show that $\{x_n\}$ fails to be Cauchy by showing that $\frac{1}{2} \leq x_{2n} - x_n$.