THE URYSOHN UNIVERSAL SPACE AND HYPERCONVEXITY

ASUMAN GÜVEN AKSOY AND ZAIR IBRAGIMOV

Abstract. In a paper published posthumously, Pavel Samuilovich Urysohn constructed a complete, separable metric space that contains an isometric copy of every complete separable metric space. In this paper we prove that the Urysohn universal space is hyperconvex.

1. Introduction

The notion of $\delta$-hyperbolic metric spaces has come to play an important role in geometric function theory and in more recent theory of quasiconformal analysis on metric spaces. For example, this notion plays a central role in the quasisymmetric uniformization problem posed by M. Bonk ([5]). A well-known example of a $\delta$-hyperbolic metric in geometric function theory is the quasihyperbolic metric introduced by F.W Gehring and his students ([10, 11]) and later studied by several authors on general metric space settings ([6, 15]). Furthermore, M. Bonk and O. Schramm proved that every $\delta$-hyperbolic metric space is isometrically contained in a complete, geodesic, $\delta$-hyperbolic metric space ([7, Theorem 4.1]). Their proof is constructive in the sense that given a $\delta$-hyperbolic space, they add a single point to it without changing its hyperbolicity constant $\delta$ and then use transfinite induction to arrive to the desired space. A similar space can be obtained by using the notion of hyperconvexity. Namely, A. Dress ([8]) showed that the hyperconvex hull of a $\delta$-hyperbolic metric space is geodesic and $\delta$-hyperbolic (see also [9, 17]). It is also known that hyperconvex hulls are complete ([16]). There is a common thread to these constructions and it seems to us that the origins of such constructions go back to the work of Urysohn who constructed a complete, separable metric space that contains an isometric copy of every complete separable metric space ([22]). In
this context it is natural to investigate the relationship between hyperconvexity and the Urysohn universal space. It is our hope that Urysohn’s ideas will be of interest to researchers in geometric function theory. For example, does there exist a universal $\delta$-hyperbolic metric space containing an isometric copy of every $\delta$-hyperbolic space? It is worth noting that Urysohn’s ideas have been extensively explored in geometry and topology (see, for example, [13, 14, 18, 21, 23]).

2. Preliminaries

It is well known that every separable metric space $X$ isometrically embeds in a Banach space $l^\infty$ (Fréchet embedding). A theorem of Banach [3] also provides a concrete separable target for all separable spaces, namely every separable metric space embeds isometrically in $C[0,1]$. Here $C[0,1]$ is the separable Banach space of continuous real-valued functions on the closed unit interval equipped with the sup norm. However, the interest of the Urysohn space, $U$, does not lie in its universality alone; it has the following finite transitivity property: every isometry between finite subsets of $U$ extends to an isometry of $U$ onto itself. For this reason $U$ is called Urysohn universal.

**Theorem 2.1** ([22]). Let $X$ be a separable and complete metric space that contains an isometric image of every separable metric space. Then $X$ is Urysohn universal if and only if it has the finite transitivity property.

In [22] Urysohn also proved that $U$ is unique, up to an isometry. It is worth remarking that the Banach space $C[0,1]$ cannot be Urysohn universal since every isometric bijection between Banach spaces is an affine map (see [4], page 341). Urysohn’s original construction of $U$ is published in full details in [22]. For the sake of completeness and accessibility we shall briefly go through the construction. An alternative description of the Urysohn universal space is given in [12, p. 20].

**Urysohn’s construction**

Urysohn first constructs a countable metric space $U_0$ containing the image of every countable metric space for which the distance between any two points is rational. The metric completion of $U_0$, (i.e., the unique complete metric space that contain $U_0$ as a dense subset) is then the Urysohn universal space $U$. We now proceed to the construction of the space $U_0$. We start with an arbitrary countable set $U_0 = \{a_1, a_2, \cdots, a_n, \cdots \}$ and define an appropriate metric on it. To
define such a metric, Urysohn first considers the collection of all nonempty finite subsets of positive rational numbers. Denote this collection by $Q$ and enumerate it as follows. First, consider all the elements of $Q$ that consist of only one rational number and enumerate them using all the natural numbers that are not divisible by 4. Now for each $p > 1$, consider all the elements of $Q$ that consist of $p$ rational numbers and enumerate them using all the natural numbers divisible by $2^p$, but not divisible by $2^{p+1}$. In this way, every element of $Q$ receives a unique label $Q_n$ for some natural number $n$. Thus, we obtain

$$Q = \{Q_1, Q_2, \ldots, Q_n, \ldots\}.$$

For example, $Q_1, Q_2$ and $Q_3$ are single rational numbers; $Q_4, Q_{12}$ and $Q_{20}$ consist of two rational numbers; $Q_8, Q_{24}$ and $Q_{40}$ consist of three rational numbers and so on. Hence each $Q_n$ can be written in the form

$$Q_n = [r_1^{(n)}, r_2^{(n)}, \ldots, r_p^{(n)}]$$

where $r_1^{(n)}, r_2^{(n)}, \ldots, r_p^{(n)}$ are the rational elements of $Q_n$. It is clear that $p_1 = 1$ and $p_n < n$, where $p_n$ is the cardinality of $Q_n$. The metric on $U_0$ is defined in the following way. We begin by setting $\rho(a_1, a_1) = 0$. Suppose for all $i, k < n + 1$, the nonnegative value $\rho(a_i, a_k)$ are defined. For $i \leq p_n$, consider the following two cases:

**Case 1.** At least one of the inequalities

$$|r_i^{(n)} - r_k^{(n)}| \leq \rho(a_i, a_k) \leq r_i^{(n)} + r_k^{(n)} \quad \text{where } i, k \leq p_n$$

is not satisfied. Urysohn calls such $Q_n$ to be *incorrectly defined*. In this case, we define

$$\rho(a_{n+1}, a_j) = \max_{i, k \leq p_n} \rho(a_i, a_k),$$

for all $j \leq n$.

**Case 2.** All of inequalities in (2.2) are satisfied. Urysohn calls such $Q_n$ to be *correctly defined*. In this case, we define

$$\rho(a_{n+1}, a_j) = \min_{\lambda \leq p_n} \{\rho(a_j, a_\lambda) + r_\lambda^{(n)}\}, \quad \text{for all } j \leq n.$$

Urysohn shows that the distance function $\rho$ is indeed a metric and that $U_0$ is universal space for all countable metric spaces having rational values for their metrics. In [22] Urysohn proves many interesting properties of his space $(U, \rho)$. The following theorem will be used throughout this paper ([22, Theorem I]).
Theorem 2.3. Given any finite subset $x_1, x_2, \ldots, x_n$ of $U$ and any positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfying $|\alpha_i - \alpha_j| \leq \rho(x_i, x_j) \leq \alpha_i + \alpha_j$ for all $i, j \leq n$, there exists $y \in U$ such that $\rho(y, x_i) = \alpha_i$ for every $i = 1, 2, \ldots, n$.

Remark 2.4. Although it is not clear from Urysohn’s construction, the sets of the form $\{r, r, r, \ldots, r\}$ and $\{r\}$ should be considered to be the same. Indeed, the following possibility illustrates that they must be considered the same. Let $Q_1 = \{2\}$, $Q_2 = \{3\}$, $Q_3 = \{4\}$ and $Q_4 = \{1/2, 1/2\}$. If such $Q_4$ was allowed to be considered as having two elements, then according to Urysohn’s definition $Q_4$ would be incorrectly defined. In this case, $\rho(a_5, a_j) = 2$ for all $j = 1, 2, 3, 4$ and $\rho(a_3, a_4) = 7$, contradicting to the triangle inequality for $a_3, a_4, a_5$.

3. Hyperconvexity of the Urysohn universal space

We begin with the definition of hyperconvexity. Throughout this section we denote by $B(x, r)$ the closed ball centered at $x$ with radius $r$.

Definition 3.1. A metric space $(X, d)$ is said to be hyperconvex if

$$\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$$

for every collection $B(x_i, r_i)$ of closed balls in $X$ for which $d(x_i, x_j) \leq r_i + r_j$.

The notion of hyperconvexity was first introduced by Aronszajn and Panitchpakdi in [2], where it was shown that a metric space is hyperconvex if and only if it is injective with respect to nonexpansive (1-Lipschitz) mappings. Later Isbell [16] showed that every metric space has an injective hull, which is the minimal hyperconvex space containing the given space as an isometric subspace. Hyperconvex metric spaces are complete and connected [1]. The simplest examples of hyperconvex spaces are the set of real numbers $\mathbb{R}$, or a finite-dimensional real Banach space endowed with the maximum norm. While the Hilbert space $l_2$ fails to be hyperconvex, the spaces $L^\infty$ and $l^\infty$ are hyperconvex. In [1] it is also shown that there is a general “linking construction” yielding hyperconvex spaces.

Definition 3.2. A metric space $(X, d)$ is said to be metrically convex if for any $x, y \in X$ and for any $r, s > 0$

$$B(x, s) \cap B(y, t) \neq \emptyset \quad \text{if and only if} \quad d(x, y) \leq s + t.$$

Lemma 3.3. The Urysohn universal space $(U, \rho)$ is metrically convex.
Proof. Let \( x_1, x_2 \in \mathbb{U} \) and \( r_1, r_2 > 0 \) be given. Clearly, \( B(x_1, r_1) \cap B(x_2, r_2) \neq \emptyset \) implies \( \rho(x_1, x_2) \leq r_1 + r_2 \) by the triangle inequality. Thus we only need to show the reverse implication. Assume that \( \rho(x_1, x_2) \leq r_1 + r_2 \).

Case 1. Suppose that \( r_1 \leq \rho(x_1, x_2) + r_2 \) and \( r_2 \leq \rho(x_1, x_2) + r_1 \). Then by Theorem 2.3 there exist \( y \in \mathbb{U} \) such that \( \rho(y, x_i) = r_i \) for \( i = 1, 2 \). Thus, \( y \in B(x_1, r_1) \cap B(x_2, r_2) \), so \( \bigcap_{i=1}^{2} B(x_i, r_i) \neq \emptyset \).

Case 2. Suppose that either \( r_1 > \rho(x_1, x_2) + r_2 \) or \( r_2 > \rho(x_1, x_2) + r_1 \). Without loss of generality we can assume that \( r_1 > \rho(x_1, x_2) + r_2 \). Then, for any \( y \in B(x_2, r_2) \) we have

\[
\rho(y, x_1) \leq \rho(y, x_2) + \rho(x_2, x_1) < r_2 + r_1 - r_2 = r_1.
\]

Hence \( y \in B(x_1, r_1) \), implying \( B(x_2, r_2) \subset B(x_1, r_1) \) and \( \bigcap_{i=1}^{2} B(x_i, r_i) \neq \emptyset \). \qed

**Definition 3.4.** We say that a metric space \((X, d)\) satisfies the finite ball intersection property if for any \( x_1, x_2, \ldots, x_n \in X \) and for any positive real numbers \( r_1, r_2, \ldots, r_n \) with \( d(x_i, x_j) \leq r_i + r_j \) we have \( \bigcap_{i=1}^{n} B(x_i, r_i) \neq \emptyset \).

**Lemma 3.5.** The Urysohn universal space \((\mathbb{U}, \rho)\) satisfies the finite ball intersection property.

Proof. Suppose that \( x_1, x_2, \ldots, x_n \in X \) and \( r_1, r_2, \ldots, r_n \) are given and suppose that \( d(x_i, x_j) \leq r_i + r_j \). Notice that if \( n = 2 \), then the claim follows from Lemma 3.3.

Case 1. Suppose that \( |r_i - r_j| \leq \rho(x_i, x_j) \). Then by Theorem 2.3 there exists \( y \in \mathbb{U} \) such that \( \rho(y, x_i) = r_i \). Hence \( y \in \bigcap_{i=1}^{n} B(x_i, r_i) \neq \emptyset \).

Case 2. Suppose that there exist \( i_0 \) and \( j_0 \) such that \( |r_{i_0} - r_{j_0}| > \rho(x_{i_0}, x_{j_0}) \). Then \( r_{i_0} > \rho(x_{i_0}, x_{j_0}) + r_{j_0} \) or \( r_{j_0} > \rho(x_{i_0}, x_{j_0}) + r_{i_0} \). Without loss of generality we can assume that \( r_{i_0} > \rho(x_{i_0}, x_{j_0}) + r_{j_0} \). Thus, as in case b) of the proof of Lemma 3.3, we obtain \( B(x_{j_0}, r_{j_0}) \subset B(x_{i_0}, r_{i_0}) \). Therefore, it is enough to show that

\[
\bigcap_{\{1, 2, \ldots, n\} \setminus \{i_0\}} B(x_i, r_i) \neq \emptyset
\]
given that \( \rho(x_i, x_j) \leq r_i + r_j \) for all \( i, j \in \{1, 2, \ldots, n\} \setminus \{i_0\} \).
Repeating the above procedure for the points \(x_1, x_2, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_n \in X\) and for numbers \(r_1, r_2, \ldots, r_{i_0-1}, r_{i_0+1}, \ldots, r_n\), we obtain, just like in the cases a) and b) above, that either the balls \(B(x_i, r_i), i = \{1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, n\}\), have a nonempty intersection or one ball is contained in another. Continuing in this fashion we can reduce the proof to the case of two balls which, as noted above, follows from Lemma 3.3. The proof is complete. □

**Theorem 3.6.** The Urysohn universal space \((U, \rho)\) is hyperconvex.

**Proof.** We need to show that for any family \(\{x_\alpha\}, \alpha \in \mathcal{I}\), of points in \(U\) and any family of positive numbers \(\{r_\alpha\}\) with the property \(\rho(x_\alpha, x_\beta) \leq r_\alpha + r_\beta\) the intersection \(\bigcap_{\alpha \in \mathcal{I}} B(x_\alpha, r_\beta)\) is nonempty. If the collection \(\{x_\alpha\}\) is finite the claim follows from Lemma 3.5. Suppose \(\{x_\alpha\}\) is countable. By re-labeling we put \(\{x_1, x_2, \ldots, x_n, \ldots\}\) and \(\{r_1, r_2, \ldots, r_n, \ldots\}\).

Fix \(n \geq 1\) and consider the finite subcollection \(\{x_1, x_2, \ldots, x_n\}\). By Lemma 3.5 there exists \(y_n \in U\) such that \(y_n \in \bigcap_{k=1}^{n} B(x_k, r_k)\). In particular, \(y_n \in B(x_1, r_1)\).

Now consider the sequence \(\{y_n\}\) in \(U\). Since \(U\) is complete and \(B(x_1, r_1)\) is closed, \(B(x_1, r_1)\) is compact. Therefore, there exists a subsequence \(\{y_{n_k}\}\) of \(\{y_n\}\) converging to a point \(y \in B(x_1, r_1)\). We claim that \(y \in \bigcap_{n=1}^{\infty} B(x_n, r_n)\).

Suppose that \(y \notin \bigcap_{n=1}^{\infty} B(x_n, r_n)\). Then there exists \(n_0\) such that \(y \notin B(x_{n_0}, r_{n_0})\). Since \(B(x_{n_0}, r_{n_0})\) is closed, there exists \(\epsilon > 0\) such that

\[B(y, \epsilon) \cap B(x_{n_0}, r_{n_0}) = \emptyset.\]

Since \(B(x_{n_0}, r_{n_0})\) is closed, there exists \(\epsilon > 0\) such that

\[B(y, \epsilon) \cap B(x_{n_0}, r_{n_0}) = \emptyset.\]

Since \(n_0 \leq n_k\), we obtain the required contradiction. Thus, \(y \in \bigcap_{n=1}^{\infty} B(x_n, r_n)\).

Suppose now that the family \(\{x_\alpha\}\) is uncountable. Recall that a topological space is said to be Lindelöf if every open cover has a countable subcover and that a metric space is Lindelöf if and only if it is separable (see, for example, [19, p. 192]). Thus, the Urysohn universal space \(U\) is Lindelöf and hence every open cover
of \( U \) has a countable subcover. We will show that \( \bigcap_{\alpha \in I} B(x_\alpha, r_\alpha) \neq \emptyset \). Assume that \( \bigcap_{\alpha \in I} B(x_\alpha, r_\alpha) = \emptyset \). Then \( U \setminus \bigcap_{\alpha \in I} B(x_\alpha, r_\alpha) = U \). By the De Morgan formulas, 
\[ U = U \setminus \bigcap_{\alpha \in I} B(x_\alpha, r_\alpha) = \bigcup_{\alpha \in I} (U \setminus B(x_\alpha, r_\alpha)) \]
Since \( U \setminus B(x_\alpha, r_\alpha) \) is open, the collection \( \{U \setminus B(x_\alpha, r_\alpha), \alpha \in I\} \) is an open cover of \( U \). Since \( U \) is Lindelöf, there exists a countable subcover, say \( \{U \setminus B(x_1, r_1), U \setminus B(x_2, r_2), \ldots, U \setminus B(x_n, r_n), \ldots\} \).
That is, 
\[ U = \bigcup_{k=1}^{\infty} (U - B(x_k, r_k)) \]
Again, by De Morgan we have 
\[ U = \bigcup_{k=1}^{\infty} (U - B_k) = U - \bigcap_{k=1}^{\infty} B(x_k, r_k) \] 
so that \( \bigcap_{k=1}^{\infty} B(x_k, r_k) = \emptyset \). However, since \( \rho(x_i, x_j) \leq r_i + r_j \), by the countable case above we obtain \( \bigcap_{k=1}^{\infty} B(x_k, r_k) \neq \emptyset \), a contradiction. \( \square \)

References


Asuman Güven Aksoy  
Claremont McKenna College  
Department of Mathematics  
Claremont, CA 91711, USA  
E-mail: aaksoy@cmc.edu

Zair Ibragimov  
California State University, Fullerton  
Department of Mathematics  
Fullerton, CA, 92831, USA  
E-mail: zibragimov@fullerton.edu