Mixed Partial Derivatives and Fubini's Theorem

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Introduction

A most fascinating aspect of calculus is its power to surprise even an experienced mathematician. Just when it appears that all ideas, results and connections have been discovered and thoroughly analyzed, the horizon suddenly broadens and somebody cries the familiar "eureka". The reason could be either a new result, a simpler way to prove an existing theorem, or a previously missed connection between different ideas. This potential for enrichment is second to none, and it reaffirms the unparalleled educational value of this area of mathematics.

The purpose of this paper is to provide one more piece of evidence in support of this vibrant vitality of calculus by proving a result that links together two theorems, one considered by many intuitive, while the other is certainly nonintuitive. The first is Fubini's theorem on exchanging the order of integration. The nonintuitive result is the equality of mixed partial derivatives. The link between the two results is their equivalence, which is established in Theorem 3. For the sake of simplicity we confine our discussion to functions defined on open sets of the plane and to second order mixed partial derivatives. The interested reader can easily extend the obtained result to more general situations. All functions of two variables are assumed to be continuous ("jointly continuous" is the terminology preferred by some authors).

Marsden and Hoffman [3] use the Mean Value Theorem to give a plausible proof of the equality $f_{xy} = f_{yx}$, while emphasizing its nonintuitive nature. Kaplan [2] derives the same equality from Fubini's Theorem in the case when f, f_x, f_y, f_{yx}, and f_{xy} are continuous.

Fubini's Theorem and the equality of the mixed partial derivatives

Let U be an open subset of $\mathbf{R}^2$. $C(U, \mathbf{R})$ denotes the vector space of real continuous functions on U, and $C^1(U, \mathbf{R})$ the subspace of $C(U, \mathbf{R})$ of those functions f such that $f_x, f_y \in C(U, \mathbf{R})$. Our goal is to prove (see Theorem 3) that the following results are equivalent.

(i) Let $g \in C(U, \mathbf{R})$ and $[a, b] \times [c, d] \subset U$. Then (Fubini's Theorem)

$$\int_a^b \int_c^d g(x, y)\, dy\, dx = \int_c^d \int_a^b g(x, y)\, dx\, dy \tag{1}$$

(ii) Assume that $f \in C^1(U, \mathbf{R})$ and $f_{xy} \in C(U, \mathbf{R})$. Then f_{yx} exists and $f_{yx} = f_{xy}$ in U.

The symbol f_{xy} denotes the second partial derivative of f, first with respect to x and then with respect to y. Three comments are in order.

Fubini's Theorem not only states that the two iterated integrals are equal to each other, but also that they are equal to the double integral of g over the rectangle $[a, b] \times [c, d]$. Although in (i) only the equality of the two iterated integrals is needed, we have decided to call (1) "Fubini's Theorem" since the double integral of g over the rectangle $[a, b] \times [c, d]$ is equal to the two iterated integrals.

It was known to Cauchy [4] that the integration of a real-valued continuous function on a rectangle $[a, b] \times [c, d]$ could be reduced to two successive integrations: first on $[c, d]$ and then on $[a, b]$. An extension of this result to measurable bounded functions was obtained by Lebesgue (1902). In 1907 Guido Fubini, one of Italy's most productive and eclectic mathematicians, proved the theorem for Lebesgue integrable functions defined on $A \times B$, the Cartesian product of two measurable sets.

Almost all calculus textbooks replace (ii) with the more familiar form

(ii′) Let $f \in C^1(U, \mathbf{R})$. Assume that $f_{yx}, f_{xy} \in C(U, \mathbf{R})$. Then $f_{yx} = f_{xy}$ in U.

The version we propose is more general [3] and appropriate for our purposes.

The equivalence between (i) and (ii) will be established using the two most powerful theorems of integral calculus. They are stated below without proof.

Theorem 1 (Fundamental Theorem of Integral Calculus). *Let $f : [a, b] \to \mathbf{R}$ be continuous. Define*

$$F(x) = \int_a^x f(s)\, ds$$

Then F is continuous in $[a, b]$, differentiable in (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$.

Theorem 2 (Fundamental Formula of Integral Calculus). *Let $G, f : [a, b] \to \mathbf{R}$ be continuous. Assume that $G'(x) = f(x)$ for all $x \in (a, b)$. Then*

$$\int_a^b f(x)\, dx = G(b) - G(a). \tag{2}$$

Theorem 1 is derived from the Extreme Value Theorem. The proof of Theorem 2 can be obtained from the equality (see Theorem 1)

$$\int_a^b f(x)\,dx = F(b). \tag{3}$$

combined with a corollary of the Mean Value Theorem. The corollary states that $f - g$ is constant if $f, g : [a, b] \to \mathbf{R}$ are continuous in $[a, b]$, and differentiable in (a, b) with equal derivatives. Thus, the Fundamental Theorem and the Fundamental Formula of Integral Calculus follow from a remarkable property of continuous functions (the Extreme Value Theorem) and a beautiful geometric property of differentiable functions (the Mean Value Theorem). These results, in turn, go back to one of the most intuitive and powerful principles: the Nested Interval Principle (see [**1**] for a detailed approach). Given these preliminaries, it is not surprising that Theorems 1 and 2 have far reaching consequences and applications.

In addition to Theorems 1 and 2 we shall need the following lemma.

Lemma 1. *Let $g \in C(I \times J, \mathbf{R})$ and $(a, c) \in I \times J$ where I and J are non-empty open intervals. Then the functions*

$$f_1(x, y) = \int_a^x g(u, y)\,du \quad \text{and} \quad f_2(x, y) = \int_c^y g(x, v)\,dv$$

belong to $C(I \times J, \mathbf{R})$.

Proof. Assume that $x, x + h \in I$, and $y, y + k \in J$. Theorem 1 and the Mean Value Theorem give

$$\begin{aligned} f_1(x + h, y + k) &= f_1(x, y + k) + g(x + th, y + k)h \\ &= \int_a^x g(u, y + k)\,du + g(x + th, y + k)h \end{aligned}$$

for some $t = t(h) \in (0, 1)$. The continuity of g implies that as $(h, k) \to (0, 0)$ the last term of the above equality goes to $g(x, y) \times 0 = 0$ and the second to the last goes to $\int_a^x g(u, y)\,du = f_1(x, y)$. Hence $f_1(x, y)$ is continuous. The continuity of f_2 is obtained in a similar manner. ■

We have now the background needed for proving the equivalence between Fubini's Theorem and the equality of the mixed partial derivatives.

Theorem 3. *Propositions (i) and (ii) are equivalent.*

Proof. To see that (i) $\Rightarrow$ (ii), let $(x, y) \in U$. Since U is open we can find $r > 0$ such that the open disk centered at (x, y) and with radius r is contained in U. Select (a, c) in the disk. The continuity of f_x, f_y, and f_{xy} together with Theorem 2 imply that

$$\int_a^x \int_c^y f_{xy}(u, v)\,dv\,du = f(x, y) - f(x, c) - f(a, y) + f(a, c).$$

By Fubini's Theorem we can exchange the order of integration on the left-hand-side. We obtain

$$\int_c^y \int_a^x f_{xy}(u, v)\,du\,dv = f(x, y) - f(x, c) - f(a, y) + f(a, c). \tag{4}$$

Differentiate both sides of (4), first with respect to y and then with respect to x. By Theorem 1 the left-hand-side gives f_{xy}. The right-hand-side gives f_{yx}. Hence $f_{yx} = f_{xy}$ in U.

To see that (ii) $\Rightarrow$ (i), let $g \in C(U, \mathbf{R})$ and $Q = [a, b] \times [c, d]$, $a < b$, $c < d$, be a rectangle in U. There are bounded open intervals I, J such that $Q \subset I \times J = V \subset U$. Define $h, f : V \to \mathbf{R}$ by

$$h(x, y) = \int_c^y g(x, v)\, dv, \qquad f(x, y) = \int_a^x h(u, y)\, du.$$

Theorem 1 implies that $f_x = h$ in V. From Lemma 1 it follows that $f_x \in C(V, \mathbf{R})$. Again from Theorem 1 we derive that $f_{xy} = g$ in V. Hence, $f_{xy} \in C(V, \mathbf{R})$. To apply (ii) to f we also need $f_y \in C(V, \mathbf{R})$. We obtain this result in a straightforward manner by using Theorem 1 and the Mean Value Theorem. Assume that $r > 0$. For $r < 0$ the proof is similar. Then

$$f(x, y + r) - f(x, y) = \int_a^x (h(u, y + r) - h(u, y))\, du.$$

By Theorem 1, h is differentiable with respect to y and $h_y(x, y) = g(x, y)$. The Mean Value Theorem applied to h in $[y, y + r]$ gives

$$h(u, y + r) - h(u, y) = h_y(u, y + tr)r = g(y, y + tr)r$$

for some $t = t(r) \in (0, 1)$. Therefore

$$\frac{f(x, y + r) - f(x, y)}{r} = \int_a^x g(u, y + tr)\, du.$$

As $r \to 0$, $t(r) \to 0$ and the continuity of g implies that $f_y(x, y) = \int_a^x g(u, y)\, du$. Thus, by Lemma 1, $f_y \in C(V, \mathbf{R})$. All assumptions of (ii) are now satisfied. Hence, f_{yx} exists and $f_{yx} = f_{xy}$ in V. A straightforward application of Theorem 2 to f_{xy} and f_{yx} yields

$$\int_a^b \int_c^d f_{xy}(x, y)\, dy\, dx = \int_c^d \int_a^b f_{xy}(x, y)\, dx\, dy.$$

Since $f_{yx} = f_{xy} = g$ in V, the order of integration can be reversed. ∎

We conclude with two remarks.

1. As mentioned previously, Fubini's Theorem is more intuitive, while the equality of the mixed partials is not easy to visualize. See, for example [3, p. 358] for a discussion on this matter. Thus, Theorem 3 gives the opportunity to show that an intuitive property implies a result which is not as transparent.
2. With a strategy similar to the one used in the proof of Theorem 3 one can show the equivalence between other results: for example the product formula for derivatives and the formula of integration by parts; the equality of mixed partials and Leibniz's formula for differentiation under the integral sign etc.

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(*Amer. Math. Monthly* **68** (1961) 56–57). Our Theorem 3 differs from Seeley's result since we only require $f \in C^1(U, \mathbf{R})$ and $f_{xy} \in C(U, \mathbf{R})$. We derive the existence and continuity of f_{yx}, together with the equality $f_{xy} = f_{yx}$ from Fubini's Theorem and without using Leibniz's rule.

References

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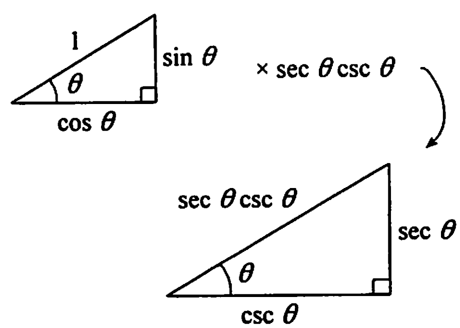
4. A. C. Zaanen, *Linear Analysis*, North Holland, 1953.

Mathematics Without Words

Need a solution to $x + y = xy$? Roger Nelsen (Lewis & Clark College, nelsen@lclark.edu) shows how Pythagoras can supply one:



$$\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta.$$