

Characterization Conditions and the Numerical Index

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Abstract. In this paper we survey some recent results concerning the numerical index $n(\cdot)$ for large classes of Banach spaces, including vector valued ℓ_p -spaces and ℓ_p -sums of Banach spaces where $1 \leq p < \infty$. In particular by defining two conditions on a norm of a Banach space X , namely a Local Characterization Condition (LCC) and a Global Characterization Condition (GCC), we are able to show that if a norm on X satisfies the (LCC), then $n(X) = \lim_m n(X_m)$. For the case in which \mathbb{N} is replaced by a directed, infinite set S , we will prove an analogous result for X satisfying the (GCC). Our approach is motivated by the fact that $n(L_p(\mu, X)) = n(\ell_p(X)) = \lim_m n(\ell_p^m(X))$ [12].

1 Introduction

Let X be a Banach space over \mathbb{R} or \mathbb{C} . We write B_X for the closed unit ball and S_X for the unit sphere of X . The dual space is denoted by X^* and the Banach algebra of all continuous linear operators on X is denoted by $B(X)$. For a linear subspace Y of X we denote by $\mathcal{P}(X, Y)$ the set of all linear, continuous projections from X onto Y .

Definition 1.1. The *numerical range* of $T \in B(X)$ is defined by

$$W(T) = \{x^*(Tx) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

The *numerical radius* of T is then given by

$$\nu(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

Clearly, $\nu(\cdot)$ is a semi-norm on $B(X)$ and $\nu(T) \leq \|T\|$ for all $T \in B(X)$. The *numerical index* of X is defined by

$$n(X) = \inf\{\nu(T) : T \in S_{B(X)}\}.$$

Equivalently, the numerical index $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq \nu(T)$ for every $T \in B(X)$. The concept of numerical index was first introduced by Lumer [25] in 1968. Since then, much attention has been paid to this equivalence constant between the numerical radius and the usual norm in the Banach algebra of all bounded linear operators of a Banach space. It turns out that numerical index behave differently with

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respect to real or complex Banach spaces. In particular, it is known that $0 \leq n(X) \leq 1$ if X is a real space, and $\frac{1}{e} \leq n(X) \leq 1$ if X is a complex space [11]. Furthermore, $n(X) > 0$ if and only if $\nu(\cdot)$ and $\|\cdot\|$ are equivalent norms. For a Hilbert space H of dimension greater than one, it is known that $n(H) = \frac{1}{2}$ in the complex case and $n(H) = 0$ in the real case. Classical references for this subject are the monographs by F. Bonsall and J. Duncan [5], [6] and the book of K. E. Gustafson and D. K. Rao [18] for the Hilbert space case. For more recent results relating to our discussion in this paper we refer the reader to [2], [3], [12], [13], [15], [18], [20] [24], [31], [33] and [34]. The last detailed survey paper on the numerical index appeared in 2006 by V. Kadets, et al, [21], which is a good source for open problems and references pertaining to the numerical index. In this paper we survey results on numerical index emphasizing progress since 2006. However, our focus is motivated by results such as the one given in [12] where the numerical index of vector-valued function spaces is considered and a proof of

$$n(L_p(\mu, X)) = \lim_m n(\ell_p^m(X))$$

is provided for a Banach space X and for $1 \leq p < \infty$. In [4] above type of limit theorem for a class of Banach spaces including vector valued ℓ_p or L_p spaces are obtained.

The study of the numerical index of absolute sums of Banach spaces is given in [33], where under suitable conditions it is shown that the numerical index of a sum is greater or equal to the lim sup of the numerical index of the summands (see Theorem 5.1 of [33]). Here we present a result which is an improvement over the limit theorem presented in [33]. In [4], we show the lim inf of the numerical index of the summands is greater or equal to the numerical index of the sum provided Banach space satisfies certain conditions.

We discuss what we mean by the norm of a Banach space satisfying a condition called the Local Characterization Condition (LCC) or a condition called the Global Characterization Condition (GCC) and provide examples of Banach spaces satisfying the above mentioned conditions. We show if a norm on X satisfies the local characterization condition, then

$$n(X) = \lim_m n(X_m) \quad \text{and} \quad n(X) = \lim_{s \in S} n(X_s)$$

with the second equality holding true when X satisfies the (GCC) and where S is any directed, infinite set.

2 Characterization conditions and numerical index

Given an arbitrary family of $\{X_i \mid i \in I\}$ of Banach spaces, we denote $[\oplus_{i \in I} X_i]_{c_0}$ (resp. $[\oplus_{i \in I} X_i]_{\ell_1}$, $[\oplus_{i \in I} X_i]_{\ell_\infty}$), the c_0 -sum, (resp. ℓ_1 -sum, ℓ_∞ -sum) of the family. In the case when I consists of only two elements, we use the notation $X \oplus_\infty Y$ or $X \oplus_1 Y$. It is customary to use the notation $c_0(X)$, $\ell_1(X)$ or $\ell_\infty(X)$ for countable copies of X . First task of investigation is to check whether or not numerical index of c_0 , ℓ_1 , and ℓ_∞ -sums can

be computed in terms of the summands. The following proposition due to M. Martin and R. Paya ([26], Proposition 1), gives an affirmative answer.

Proposition 2.1. *Let $\{X_i \mid i \in I\}$ be a set of Banach spaces. Then*

$$n([\oplus_{i \in I} X_i]_{c_0}) = n([\oplus_{i \in I} X_i]_{\ell_1}) = n([\oplus_{i \in I} X_i]_{\ell_\infty}) = \inf_i n(X_i).$$

The above proposition is not true for ℓ_p -sums if $p \neq 1, \infty$. However, there is an interesting example given in ([26], Example 2.b) which asserts the existence of a real Banach space X for which the numerical radius is a norm but it is not equivalent to operator norm. In other words the numerical index of X is zero even though the numerical radius $\nu(T) > 0$ for every $T \in B(X)$.

For direct sums of Banach spaces, under some general conditions it is shown that the numerical index of the sum is less or equal than the infimum of the numerical indices of the summands (see [33], Theorem 2.1 and Corollary 3.1). Furthermore, they also consider the numerical index of a Banach space which contains a dense increasing union of one-complemented subspaces and prove the following:

Theorem 2.2. ([33], Theorem 5.1) *Let X be a Banach space, let I be a directed set, and $\{X_i \mid i \in I\}$ be an increasing family of one-complemented closed subspaces such that $X = \bigcup_{i \in I} X_i$. Then,*

$$n(X) \geq \limsup_{i \in I} n(X_i).$$

Later, in [4], it is proved that the \liminf of the numerical index of the summands is greater than or equal to the numerical index of the sum if the Banach space satisfies a condition called the local characterization condition (LCC) or a condition called the global characterization condition (GCC). We show if a norm on X satisfies the local characterization condition, then

$$n(X) = \lim_m n(X_m)$$

and

$$n(X) = \lim_{s \in S} n(X_s)$$

where S is any directed, infinite set and X satisfies the (GCC). We also provide examples of spaces where (LCC) or (GCC) is satisfied.

The following theorem, which is a direct consequence of ([34], Theorem 2.5), plays a crucial role in our further investigations.

Theorem 2.3. *Let X be a Banach space over \mathbb{R} or \mathbb{C} and let*

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}.$$

Denote by π_1 the natural projection from $\Pi(X)$ onto S_X defined by $\pi_1(x, x^) = x$. Fix a set $\Gamma \subset \Pi(X)$ such that $\pi_1(\Gamma)$ is dense in S_X . Then for any $T \in B(X)$,*

$$\nu(T) = \sup\{|x^*(Tx)| : (x, x^*) \in \Gamma\}.$$

Applying the above theorem we can prove:

Corollary 2.4. *Let X be an infinite-dimensional Banach space and let $Y \subseteq X$ be its linear subspace whose norm-closure is equal to X . Define for $L \in \mathcal{L}(X)$,*

$$\nu_Y(L) = \sup\{|x^*Lx| : x^* \in S_{X^*}, x \in S_Y, x^*(y) = 1\}. \quad (2.1)$$

Then $\nu(L) = \nu_Y(L)$.

Definition 2.5. Let X be a Banach space and $X_1 \subset X_2 \subset \dots \subset X$ be its subspaces such that $X = \bigcup_{m=1}^{\infty} X_m$. Suppose for any $m \in \mathbb{N}$ there exists $P_m \in \mathcal{P}(X_{m+1}, X_m)$ with $\|P_m\| = 1$. We say the norm on X , $\|\cdot\|_X$ satisfies the *Local Characterization Condition* (LCC) with respect to $\{P_m\}_{m=1}^{\infty}$ if and only if for any $m \in \mathbb{N}$ there exists D_m a dense subset of $S_{X_{m+1}}$ such that for any $x \in D_m$ there exists $x^* \in S_{X_{m+1}^*}$ a norming functional for x in X_{m+1}^* and a constant $b_m(x) \in \mathbb{R}_+$ such that $b_m(x)x^*|_{X_m}$ is a norming functional for P_mx in X_m^* . (In fact, if $P_m(x) \neq 0$, then $b_m(x) = \|P_m(x)\|/x^*(P_m(x))$.)

Now we present an example of a Banach space X satisfying the (LCC) given in Definition (2.5).

Example 2.6. Let for $n \in \mathbb{N}$ $(Y_n, \|\cdot\|_n)$ be a Banach space. Set $X_1 = Y_1$ and $X_n = X_{n-1} \oplus Y_n$. Let for $n \in \mathbb{N}$, let $p_n \in [1, \infty)$. Define a norm $|\cdot|_1$ on X_1 by $|x|_1 = \|x\|_1$ and a norm $|\cdot|_2$ on X_2 by

$$|(x_1, x_2)|_2 = (\|x_1\|_1^{p_1} + \|x_2\|_2^{p_1})^{1/p_1},$$

where $x_i \in Y_i$ for $i = 1, 2$. Then, having defined $|\cdot|_n$ for $x = (x_1, \dots, x_n) \in X_n$, we can define $|\cdot|_{n+1}$ on X_{n+1} by

$$|(x, x_{n+1})|_{n+1} = (|x|_n^{p_n} + \|x_{n+1}\|_{n+1}^{p_n})^{1/p_n}.$$

Note that if $x \in X_n$, and $m \geq n$, then $|x|_m = |x|_n$. Let

$$F = \{\{y_n\} : y_n \in Y_n \text{ and } y_n = 0 \text{ whenever } n \geq m \text{ depending on } \{y_n\}\}.$$

One can identify F with $\bigcup_{n=1}^{\infty} X_n$, thus enabling us to define, for $x \in F$, its norm as:

$$\|x\|_F = \lim_n |x|_n,$$

because for fixed $x \in F$ the sequence $|x|_n$ is constant from some point on by the above mentioned property. Notice that the completion of F (we will denote it by X) is equal to the space of all sequences $\{x_n\}$ such that $x_n \in X_n$ and

$$\lim_n \|Q_n x\|_F = \sup_n \|Q_n x\|_F < +\infty,$$

where for $n \in \mathbb{N}$ and $x = (x_1, x_2, \dots)$

$$Q_n(x) = (x_1, \dots, x_n, 0, \dots).$$

Indeed, let $\{x^s\}$ be a Cauchy sequence in X . Notice that by definition of $\|\cdot\|_F$, $\|Q_n|_X\| = 1$. Hence for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $s, k \geq N$ and $n \in \mathbb{N}$,

$$\|Q_n(x^s - x^k)\|_n \leq \epsilon.$$

Consequently, for any $n \in \mathbb{N}$, $Q_n(x^s)$ converges to some point in X_n . Hence for any $i \in \mathbb{N}$ $(x^s)_i \rightarrow x_i \in Y_i$. Set $x = (x_1, x_2, \dots)$. Then, it is easy to see that $x \in X$, since any Cauchy sequence is bounded and

$$\|Q_n(x)\|_F = \lim_s \|Q_n(x^s)\|_F \leq \sup_s \|x^s\|_F < +\infty.$$

Moreover, for fixed $\epsilon > 0$, for $s, k \geq N$ and any $n \in \mathbb{N}$,

$$\|Q_n(x^k - x^s)\|_F \leq \|x^s - x^k\|_F \leq \epsilon.$$

Hence fixing $k \geq N$ and taking limit over s we get for any $n \in \mathbb{N}$,

$$\|Q_n(x^k - x)\|_F \leq \epsilon,$$

and consequently $\|x - x^k\|_X \leq \epsilon$ for $k \geq N$, which shows that $\{x^k\}$ converges to $x \in X$. Hence X is a Banach space. Since for any $x \in X$, $\lim_n \|Q_n(x) - x\| = 0$, F is a dense subset of X . Note that, for any $n \in \mathbb{N}$, a map $P_n : X_{n+1} \rightarrow X_n$ given by

$$P_n(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n, 0),$$

is a linear projection of norm one. By Definition (2.5) and the proof from Example (2.8), the (LCC) is satisfied for the norm on X .

Definition 2.7. Let X be a Banach space and let $\{X_s\}_{s \in S}$ be a family of subspaces of X such that $X = \overline{\bigcup_{s \in S} X_s}$. Assume that for any $s_1, s_2 \in S$ there exists $s_3 \in S$ such that $X_{s_1} \cup X_{s_2} \subset X_{s_3}$, i.e. the family $\{X_s\}_{s \in S}$ forms a directed set. Suppose for any $s \in S$ there exists $P_s \in \mathcal{P}(X, X_s)$ with $\|P_s\| = 1$. We say the norm on X , $\|\cdot\|_X$ satisfies the *Global Characterization Condition* (GCC) with respect to $\{P_s\}_{s \in S}$ if and only if for any $s \in S$ there exists D_s a dense subset of S_X such that for any $x \in D_s$ there exists $x^* \in S_{X_s^*}$ a norming functional for x in X and a constant $b_m(x) \in \mathbb{R}_+$ such that $b_s(x)x^*|_{X_s}$ is a norming functional for $P_s x$ in X_s^* . (In fact, if $P_s(x) \neq 0$, then $b_s(x) = \|P_s(x)\|/x^*(P_s(x))$.)

Now we present an example of a Banach space X satisfying the condition (GCC) given in Definition(2.7).

Example 2.8. Let S be a directed and infinite set. Fix $p \in [1, \infty)$. Let $X^p = (\bigoplus_{s \in S} X_s)_p$ be the direct, generalized l^p -sum of Banach spaces $(X_s, \|\cdot\|_s)_{s \in S}$, defined as

$$X^p = \{(x_s)_{s \in S} : x_s \in X_s, \text{card}(\text{supp}((x_s)_{s \in S})) \leq \aleph_0 \text{ and } \sum_{s \in S} (\|x_s\|_s)^p < \infty\},$$

where $\text{supp}((x_s)_{s \in S}) = \{s \in X_s : x_s \neq 0\}$. Clearly, the norm of $x \in X^p$ is

$$\|x\| = \left(\sum_{s \in S} (\|x_s\|_s)^p \right)^{1/p}$$

and in case $S = \mathbb{N}$ and $X_i = X$ for all $i \in \mathbb{N}$, $X^p = \ell_p(X)$. Fix any finite set $W \subset S$. Next, consider spaces $Z_W = \oplus_{s \in W} X_s$ and the projections

$$P_W((x_s)_{s \in S}) = (z_s)_{s \in S},$$

where $z_s = x_s$ for $s \in W$ and $z_s = 0$ otherwise. Let $F = \{W \subset S : \text{card}(W) < \infty\}$. Now we show that the (GCC) is satisfied for X , $\{Z_W\}_{W \in F}$ and $\{P_W\}_{W \in F}$. It is obvious that $\|P_W\| = 1$ for any finite subset W of S and $p \in [1, \infty)$. Now assume that $1 < p < \infty$. To show that the characterization condition is satisfied for the norm on X , note that for any $x \in X^p \setminus \{0\}$ there exists a norming functional of the form

$$x^* = \frac{(\|x_s\|_s^{p-1} x_s^*(\cdot))_{s \in S}}{\left(\sum_{s \in S} (\|x_s\|_s)^p\right)^{\frac{p-1}{p}}}$$

where $x_s^* \in X_s^*$ is a norming functional for $x_s \in X_s$. Setting $C = \left(\sum_{s \in S} (\|x_s\|_s)^p\right)^{\frac{p-1}{p}}$, to see that $\|x^*\| \leq 1$, let $y \in X$ be an element with $\|y\| = 1$, then

$$|x^*(y)| = \left| \frac{\sum_{s \in S} (\|x_s\|_s)^{p-1} x_s^*(y_s)}{C} \right| \leq \frac{1}{C} \sum_{s \in S} (\|x_s\|_s)^{p-1} |x_s^*(y_s)|.$$

Applying the Hölder inequality with conjugate pairs p and q :

$$|x^*(y)| \leq \frac{1}{C} \left[\sum_{s \in S} (\|x_s\|_s)^{p-1} \right]^{\frac{1}{q}} \cdot \left[\sum_{s \in S} \|y_s\|_s^p \right]^{\frac{1}{p}}.$$

Since $q = \frac{p}{p-1}$ and $\|y\| = \left[\sum_{s \in S} \|y_s\|_s^p \right]^{\frac{1}{p}} = 1$ we have $|x^*(y)| \leq 1$. It is easy to see that x^* is a norming functional for x because

$$x^*(x) = \frac{1}{C} \sum_{s \in S} (\|x_s\|_s)^{p-1} x_s^*(x_s) = \frac{\|x\|^p}{\|x\|^{p-1}} = 1.$$

Furthermore, if $P_W \neq 0$, from

$$(P_W x)^* = \frac{(\|x_w\|_w^{p-1} x_w^*(\cdot))_{w \in W}}{\left(\sum_{w \in W} (\|x_w\|_w)^p\right)^{\frac{p-1}{p}}}$$

and writing $x^*|_{Z_W}$, we obtain that $b_W(x) = \frac{\|x\|^{p-1}}{\|P_W x\|^{p-1}}$. If $p = 1$ then for any $x \in X^1 \setminus \{0\}$ there exists a norming functional of the form $(x_s^*(\cdot))_{s \in S}$, where $x_s^* \in X_s^*$ is a norming functional for $x_s \in X_s$. It is easy to see that

$$\|(x_s^*(\cdot))_{s \in S}\| = \sup_{s \in S} \|x_s^*\|.$$

Reasoning as in the previous case we get that (GCC) is satisfied for $p = 1$.

Remark 2.9. Note that if $S = \mathbb{N}$ and $X_s \subset X_z$ for $s, z \in \mathbb{N}$, $s \leq z$ then the (GCC) implies the (LCC).

The above definition is motivated by the space $X = \ell_p$ with $1 < p < \infty$, $X_m = \ell_p^{(m)}$ and a sequence of projections $\{P_m\}_{m=1}^\infty$ defined by

$$P_m(x_1, \dots, x_m, x_{m+1}, \dots) = (x_1, \dots, x_m, 0, \dots).$$

For $x \neq 0$ and $x \in \ell_p$, the form of the norming functional is $x^* = \frac{(|x_i|^{p-1} \text{sgn}(x_i))}{\|x\|_p^{p-1}}$ and clearly

$$x^*|_{X_m} = \frac{(|x_i|^{p-1} \text{sgn}(x_i))}{\|x\|_p^{p-1}} \text{ where } i \in \{1, 2, \dots, m\}$$

and the norming functional for $P_m x$, $(P_m x)^*$ takes the form

$$(P_m x)^* = \frac{(|x_i|^{p-1} \text{sgn}(x_i))}{\|P_m x\|_p^{p-1}} \text{ where } b_m(x) = \frac{\|x\|_p^{p-1}}{\|P_m x\|_p^{p-1}}.$$

The above (GCC) is also satisfied for norms of ℓ_1 and c_0 (with the same sequence $\{P_m\}_{m=1}^\infty$).

We start by investigating first some consequences of (LCC).

Proposition 2.10. [4] Let X be a Banach space satisfying (LCC) with respect to $\{P_m\}_{m=1}^\infty$. For a fixed $m \in \mathbb{N}$ and $L \in \mathcal{L}(X_m)$, define a sequence

$$w_m(L) = \nu(L), w_{m+1}(L) = \nu(L \circ P_m), \dots, w_{m+j}(L) = \nu(L \circ Q_{m,j}),$$

where $Q_{m,j} = P_m \circ \dots \circ P_{m+j-1}$. (For $j \geq 1$ $\nu(L \circ Q_{m,j})$ denote the numerical radius of $L \circ Q_{m,j}$ with respect to X_{m+j} .) Then $\nu(L) = w_m(L) = w_{m+j}(L)$ for $j = 1, 2, \dots$

Proof. Since $X_m \subset X_{m+1}$ for any $m \in \mathbb{N}$, it is easy to see that $w_{m+j}(L)$ is an increasing sequence with respect to j , since

$$w_{m+j}(L) = \sup\{|x^* L \circ Q_{m,j} x| : x \in S_{X_{m+j}}, x^* \in S_{X_{m+j}^*}, x^*(x) = 1\}$$

$$\leq \sup\{|x^* L Q_{m,j} P_{m+j} x| : x \in S_{X_{m+j+1}}, x^* \in S_{X_{m+j+1}^*}, x^*(x) = 1\} = w_{m+j+1}(L).$$

Now we prove that $w_m = w_{m+1}$. To do this for any $x \in D_m$ select $x_x^* \in S_{X_{m+1}^*}$ satisfying the requirements of Definition(2.5). Set

$$\Gamma_m = \{(x, x_x^*) \in \Pi(X) : x \in D_m\}.$$

Note that by Def.(2.5),

$$\frac{b_m(x)}{\|P_m(x)\|} = \frac{1}{x_x^*(P_m(x))} \geq 1.$$

Hence for any $(x, x_x^*) \in \Gamma_m$,

$$|x_x^* \circ L \circ P_m x| = |(x_x^*)|_{X_m} \circ L \circ P_m x| \leq \frac{b_m(x)}{\|P_m x\|} |(x_x^*)|_{X_m} \circ L \circ P_m x|$$

$$= |(b_m(x)x_x^*)|_{X_m} L(\frac{P_m x}{\|P_m x\|})| \leq \nu(L).$$

Notice that by Def.(2.5), $\pi_1(\Gamma_m) = D_m$ and D_m is dense in $S_{X_{m+1}}$. By Theorem (2.3) applied to Γ_m and $L \circ P_m$,

$$w_{m+1}(L) = \nu(L \circ P_m) \leq \nu(L) = w_m(L)$$

and thus $w_m(L) = w_{m+1}(L)$. Induction on j results in $w_m(L) = w_{m+j}(L)$. \square

Next, we examine some properties of the numerical index when the norm of the space X satisfies the LCC. Proof of the following proposition is given in [4].

Proposition 2.11. *The following hold true:*

- a) *Let $P_j \in \mathcal{P}(X_{j+1}, X_j)$ with $\|P_j\| = 1$. For a fixed $m \in \mathbb{N}$, define projections $Q_{m,j} \in \mathcal{P}(X_{m+j}, X_m)$ as $Q_{m,j} = P_m \circ P_{m+1} \circ \cdots \circ P_{m+j-1}$. Then*

$$\lim_{j \rightarrow \infty} Q_{m,j} = Q_m$$

where $Q_m \in \mathcal{P}(X, X_m)$ with $\|Q_m\| = 1$ and $X = \overline{\bigcup_{m=1}^{\infty} X_m}$.

- b) *For a fixed $m \in \mathbb{N}$ and $L \in \mathcal{L}(X_m)$ we have*

$$w_{m+j}(L) \leq \nu(L \circ Q_m)$$

for all j , where $\nu(L \circ Q_m)$ denotes the numerical radius of $L \circ Q_m$ with respect to X .

- c) *Let X satisfy (LCC) with respect to $\{P_m\}_{m=1}^{\infty}$. Then for any $m \in \mathbb{N}$ and $L \in \mathcal{L}(X_m)$,*

$$\nu(L) = \nu(L \circ Q_m),$$

where $Q_m \in \mathcal{P}(X, X_m)$ are defined in Proposition(1).

- d) *Assume that $\|\cdot\|_X$ satisfies (LCC). Then for any $m \in \mathbb{N}$,*

$$n(X_m) \geq n(X).$$

Theorem 2.12. [4] *Let X and X_m and P_m be as in Definition (2.5). Then*

$$n(X) = \lim_m n(X_m).$$

Proof. By the above Proposition, part d) we have, $n(X_m) \geq n(X)$ for any $m \in \mathbb{N}$. Hence,

$$\liminf_m n(X_m) \geq n(X).$$

By Theorem 5.1 of [33], we already know that

$$n(X) \geq \limsup_m n(X_m),$$

which proves the equality. \square

Now we present our main theorem.

Theorem 2.13. [4] *Let X and X_s and P_s be as in Definition (2.7). Then*

$$n(X) = \lim_s n(X_s).$$

Proof. By Theorem 5.1 of [33], we already know that

$$n(X) \geq \limsup_s n(X_s).$$

Now we prove that

$$\liminf_s n(X_s) \geq n(X).$$

Fix $s \in S$ and $L \in \mathcal{L}(X_s)$. We show that $\nu(L) \geq \nu(L \circ P_s)$, where $\nu(L)$ denotes the numerical radius of L with respect to X_s and $\nu(L \circ P_s)$ denotes the numerical radius of $L \circ P_s$ with respect to X . To do that, for any $x \in D_s$ select $x_x^* \in S_{X^*}$ satisfying the requirements of Definition (2.7). Let

$$\Gamma_s = \{(x, x_x^*) \in \Pi(X) : x \in D_s\}.$$

Observe that for any $s \in S$,

$$\frac{b_s(x)}{\|P_s(x)\|} = \frac{1}{x_x^*(P_s(x))} \geq 1.$$

Note that by Def.(2.7), for any $(x, x_x^*) \in \Gamma_s$,

$$\begin{aligned} |x_x^* \circ L \circ P_s x| &= |(x_x^*)|_{X_s} \circ L \circ P_s x| \leq \frac{b_s(x)}{\|P_s x\|} |(x_x^*)|_{X_s} \circ L \circ P_s x| \\ &= |(b_s(x)x_x^*)|_{X_s} L(\frac{P_s x}{\|P_s x\|})| \leq \nu(L). \end{aligned}$$

Notice that by Def.(2.7), $\pi_1(\Gamma_s) = D_s$ and D_s is dense in S_X . By Theorem (2.3) applied to Γ_s and $L \circ P_s$, $\nu(L \circ P_s) \leq \nu(L)$, as required. Hence we get immediately that

$$n(X_s) = \inf\{\nu(L) : L \in \mathcal{L}(X_s), \|L\| = 1\} = \inf\{\nu(W) : W \in \mathcal{L}(X), \|W\| = 1\} = n(X).$$

Consequently $\liminf_s n(X_s) \geq n(X)$ and finally $\lim_s n(X_s) = n(X)$, as required. \square

3 Computing the numerical index

Let $L_p(\mu, X) = L_p(\mu)$ denote the classical Banach space of p-integrable functions f from Ω into X where (Ω, Σ, μ) is a given measure space. As usual $\ell_p(X)$ denotes the Banach space of all $x = (x_n)_{n \geq 1}$, $x_n \in X$, such that $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$. Finally $\ell_p^m(X)$ is the Banach space of finite sequences $x = (x_n)_{1 \leq n \leq m}$, $x_m \in X$ equipped with the norm $(\sum_{n=1}^m \|x_n\|^p)^{\frac{1}{p}}$.

The numerical index of some vector valued function spaces are known, as shown in the following theorem.

Theorem 3.1. ([26], [29]) *Let K be a compact Hausdorff space and μ be a positive σ -finite measure. Then*

$$n(C(K, X)) = n(L_1(\mu, X)) = n(L_\infty(\mu, X)) = n(X)$$

where by $C(K, X)$ we mean the space of X -valued continuous functions on K , and by $L_1(\mu, X)$, and $(L_\infty(\mu, X))$ we denote respectively the space of X -valued μ -Bochner-integrable functions and the space of X -valued μ -Bochner-measurable and essentially bounded functions.

One of the most important families of classical Banach spaces whose numerical indices remain unknown is the family of L_p -spaces when $p \neq 1, 2, \infty$. This stayed as an open problem till 2005 when Ed-Dari [13] and Ed-Dari and Khamsi [14] made some progress proving:

Theorem 3.2. ([13], [14])

Let $1 \leq p \leq \infty$ be fixed. Then

- a) $n(L_p[0, 1]) = n(\ell_p) = \inf\{n(\ell_p^m) : m \in \mathbb{N}\}$, and the sequence $\{n(\ell_p^m)\}_{m \in \mathbb{N}}$ is decreasing.
- b) $n(L_p(\mu)) \geq n(\ell_p)$ for every positive measure μ .
- c) In the real case

$$\frac{1}{2}M_p \leq n(\ell_p^m) \leq M_p, \text{ where } M_p = \sup_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}.$$

In 2007 Ed-Dari's result in [13] was extended for the vector valued functions [12].

Theorem 3.3. [12] *Let X be a Banach space. Then for every real number p , $1 \leq p \leq \infty$, the numerical index of the Banach space $\ell_p(X)$ is given by*

$$n(\ell_p(X)) = \lim_m n(\ell_p^m(X)).$$

Remark 3.4. *In 1970, it was shown that*

$$\{n(\ell_p^2) : 1 \leq p \leq \infty\} = [0, 1]$$

in the real case(see [11]). Using the above result of Ed-dari and Khamsi [14] one can deduce

$$\frac{M_p}{2} \leq n(\ell_p^2) \leq M_p$$

where $M_p = \sup_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}$. In [28] the authors improve the lower bound to $\max\{\frac{M}{2^{1/p}}, \frac{M}{2^{1/q}}\}$ for the numerical index of the two-dimensional real L_p -space.

Recently Martin, Meri and Popov [32] gave a lower estimate for the numerical index of the real $L_p(\mu)$ - spaces, proving the following:

Theorem 3.5. *Let $p \neq 2$ and μ any positive measure. Then $n(L_p(\mu)) > 0$ in the real case.*

The idea of the proof of the above theorem is to define a new seminorm on $\mathcal{L}(L_p(\mu))$, which is in between the numerical radius and the operator norm i.e., the numerical radius and the operator norm are equivalent on $\mathcal{L}(L_p(\mu))$ for every $p \neq 2$ and every positive measure μ . This also answers the question posed in [21].

Given an operator $T \in \mathcal{L}(L_p(\mu))$, they define the *absolute numerical radius* of T to be

$$|\nu|(T) := \sup\left\{\int_{\Omega} |x^{\sharp}Tx|d\mu : x \in L_p(\mu), \|x\|_p = 1\right\} = \sup\left\{\int_{\Omega} |x|^{p-1}|Tx|d\mu : x \in L_p(\mu), \|x\|_p = 1\right\}$$

where for any $x \in L_p(\mu)$, by x^{\sharp} we mean $x^{\sharp} = |x|^{p-1}\text{sign}(x)$ for the real case and $x^{\sharp} = |x|^{p-1}\text{sign}(\bar{x})$ for the complex case. Note that with the above notation, for $x \in L_p(\mu)$ we can write:

$$\nu(T) = \sup\left\{\left|\int_{\Omega} x^{\sharp}Tx d\mu\right| : x \in L_p(\mu), \|x\|_p = 1\right\}$$

clearly $\nu(T) \leq |\nu|(T) \leq \|T\|$ and for positive operators on $L_p(\mu)$ the numerical radius and the absolute numerical radius coincide. Using the absolute numerical radius, it is proved in [32] that:

$$\nu(T) \geq \frac{M_p}{6}|\nu|(T) \quad \text{and} \quad |\nu|(T) \geq \frac{n(L_p^{\mathbb{C}}(\mu))}{2}\|T\|$$

where $M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + tp}$ and $n(L_p^{\mathbb{C}}(\mu))$ denote the complex numerical index of $L_p(\mu)$. Since $n(X) \geq \frac{1}{e}$ for any complex space they obtain

$$n(L_p(\mu)) \geq \frac{M_p}{12e}.$$

Note that $M_p > 0$ for $p \neq 2$, this extends the result that $n(\ell_p^m) > 0$ for $p \neq 2$ and $m \in \mathbb{N}$.

It is an open problem whether or not $n(L_p(\mu)) = M_p$ in the real case, and the value for $n(L_p(\mu))$ in the complex case is unknown. However using the concept of absolute numerical radius, one can define *absolute numerical index* by:

$$|n|(L_p(\mu)) := \inf\{|\nu|(T) : T \in \mathcal{L}(L_p(\mu)), \|T\| = 1\}$$

and in [27] it is shown that

Theorem 3.6. *Let $1 < p < \infty$ and let μ be a positive measure such that $\dim(L_p(\mu)) \geq 2$. Then*

$$|n|(L_p(\mu)) = \frac{1}{p^{1/p}q^{1/q}}.$$

where $q = \frac{p}{p-1}$ is the conjugate exponent to p .

Using the above theorem together with the estimate $\nu(T) \geq \frac{M_p}{6} |\nu|(T)$ given in [32], one can improve the estimation of the numerical index of $L_p(\mu)$ space in the real case by:

$$\nu(L_p(\mu)) \geq \frac{M_p}{6} \cdot \frac{1}{p^{1/p} q^{1/q}}.$$

The numerical radius of rank one operators and finite rank operators in $L_p(\mu)$ are also considered in several papers [8], [27]. The motivation to consider rank-one operators stems from several directions, one such reason being the possibility of getting more information about the geometry of the Banach space. For example a well-known theorem due to James states that a Banach space is reflexive if and only if each rank-one operator attains its norm. A version of this theorem given in [1] asserts that numerical radius characterizes reflexive Banach spaces for which rank-one operators attain their numerical radius. Additionally, rank-one operators are contained in any operator ideal, thus rank-one numerical index becomes an upper bound for any numerical index defined as associated to any operator ideal. Now we mention briefly some recent results about rank-one numerical indices.

The *rank-one numerical index* of a Banach space $n_1(X)$ is defined as:

$$n_1(X) = \inf\{\nu(T) : T \in L(X), \|T\| = 1, \dim(T(X)) \leq 1\}$$

Theorem 3.7. ([8], Theorem 2.1)

Let X be a real Banach space. Then

$$n_1(X) \geq \frac{1}{e}.$$

The fact that $n(X) \geq \frac{1}{e}$ in the complex case is known as the Bonehblust-Karlin theorem [17], and the accomplishments in the above theorems is the extension to real spaces. Furthermore, this lower bound for the rank-one numerical index is the best possible bound, as shown in [8]. Many properties of rank-one numerical indices follow from analogous ones for the classical numerical index. For example for the cases c_0 -, ℓ_1 -, or ℓ_∞ -sums we have the following:

Proposition 3.8. Let $\{X_i \mid i \in I\}$ be a of Banach spaces. Then

$$n_1([\oplus_{i \in I} X_i]_{c_0}) = n_1([\oplus_{i \in I} X_i]_{\ell_1}) = n_1([\oplus_{i \in I} X_i]_{\ell_\infty}) = \inf_i n_1(X_i).$$

It is also known that rank-one numerical index of an absolute sum of Banach spaces is less or equal than the rank-one numerical index of each summands, as stated in the following proposition.

Proposition 3.9. ([33]) Let $\{X_i \mid i \in I\}$ be a family of Banach spaces and $1 < p < \infty$. Then,

$$n_1([\oplus_{i \in I} X_i]_{\ell_p}) \leq \inf\{n_1(X_i) \mid i \in I\}$$

However, rank-one numerical indices of vector valued spaces exhibit a different behavior from that of classical numerical indices. In fact we have :

Proposition 3.10. ([8], Proposition 2.6) *Let X be a Banach space and K a compact Hausdorff space. Then, $n_1(C(K, X)) = 1$ when K is perfect and $n_1(C(K, X)) = n_1(X)$ when K is not perfect.*

Similar results also hold true for $n_1(L_1(\mu, X))$ and for $n_1(L_\infty(\mu, X))$.

Of course one can extend the definition of rank-one numerical index $n_1(X)$ to the rank- r numerical index $n_r(X)$ in a natural way by setting

$$n_r(X) := \inf\{\nu(T) : T \in L(X), \|T\| = 1, \dim(T(X)) \leq r\}$$

or to the compact numerical index by

$$n_{comp}(X) := \inf\{\nu(T) : T \in L(X), \|T\| = 1, T \text{ compact}\}.$$

Then the inequalities:

$$n_r(X) \geq n_{r+1}(X) \geq n_{comp}(X) \geq n(X)$$

follow easily. Additionally an example of a Banach space X for which $n(X) < n_{comp}(X) < n_1(X)$ also exists [8].

Remark 3.11. *Theorems 2.12 and 2.13 can be restated with the same proof for $n_r(\cdot)$ and for $n_{comp}(\cdot)$. For example, if the norm on X satisfies LCC, or equivalently when X and X_m and P_m be as in Definition (2.5), then we have:*

$$n_r(X) = \lim_m n_r(X_m).$$

Similarly, when the norm on X satisfies GCC, or equivalently, when X and X_s and P_s be as in Definition (2.7), then we have:

$$n_r(X) = \lim_s n_r(X_s).$$

where S is any directed, infinite set.

Perhaps it is also worth mentioning that if we let A be a C^* -algebra, then the numerical index of A as an algebra is 1 or 1/2 depending on whether A is commutative or not commutative [9]. Later T. Huruya [19] showed that the numerical index of A as a normed space is also 1 or 1/2 depending on A is commutative or not commutative.

Note the we do not know the value of the numerical index of the space of k -times differentiable functions on $[0, 1]$, namely $C^k[0, 1]$, or Lorentz or Orlicz spaces.

Remark 3.12. *There are many Banach spaces whose numerical indices are known, for example the following classical Banach spaces all have numerical index 1, which is the largest possible value, (see [11])*

$$c_0, c, \ell_1, \ell_\infty, C(K) \text{ for every compact } K, \text{ and } L_1(\mu)$$

all function algebras like $A(\mathbb{D})$, H^∞ (see [35]) and finite co-dimensional subspaces of $C[0, 1]$ (see [7]). Furthermore, in [11] two families of Banach spaces with numerical index 1 are identified, these are L -spaces and M -spaces.

Recall that a Banach space X is said to have numerical index 1 if and only if, the norm of T , for every $T \in B_X$ can be redefined as:

$$\|T\| = \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}. \quad (3.1)$$

The above form of the norm of T given in 3.1 has consequences on the geometry or the topology of the Banach space. For example, is it possible to re-norm a given infinite dimensional Banach space to have numerical index one? There has been a considerable amount of research done in this direction. Recall that a space X is said to be strictly convex when $\text{ext}(B_X) = S_X$ and it is well known that if X^* is strictly convex (resp. smooth) then X is smooth (resp. strictly convex), but the converse is not true. Furthermore, the norm of X is said to be Fréchet smooth when the norm of X is Fréchet differentiable at any point of S_X . There are several results which show that Banach spaces with numerical index one cannot enjoy good convexity or smoothness properties unless they are one-dimensional. In particular, such spaces have no Weakly Local Uniform Rotundity (WLUR) points in their unit ball, their norm is not Fréchet smooth, and their dual norm is neither strictly convex nor smooth. For a detailed study of these we refer the reader to [20].

Furthermore, it is well known that (see [5]), $\nu(T^*) = \nu(T)$ for every $T \in B(X)$ where T^* is the adjoint operator of T which implies that $n(X^*) \leq n(X)$ for every Banach space X , since the map $T \rightarrow T^*$ is a linear isometry from $B(X)$ into $B(X^*)$

$$n(X^*) := \inf\{\nu(S) : S \in B(X^*), \|S\| = 1\} \leq \inf\{\nu(T^*) : T \in B(X), \|T\| = 1\} = n(X)$$

for a reflexive space X , since $n(X) = n(X^{**}) \leq n(X^*)$ clearly $n(X) = n(X^*)$. Following is an example taken from [7], which proves the fact that the numerical index of the dual of a Banach space can be strictly smaller than the numerical index of the space.

Example 3.13 ([7], Example 3.1). There exists a Banach space X such that $n(X) = 1$ and $n(X^*) < 1$. where

$$X = \{(x, y, z) \in c \oplus_\infty c \oplus_\infty c : \lim x + \lim y + \lim z = 0\}.$$

Recall that a Banach space is said to be L-embedded if $X^{**} = X \oplus X_s$ for some closed subspace X_s of X^{**} . There are some recent positive results concerning numerical index of the space X and its dual X^* . For example, it is known that $n(X) = n(X^*)$ when X is L-embedded in X^{**} . If $n(X) = 1$ and X is M-embedded in X^{**} then also $n(X^*) = 1$ and $n(Y) = 1$ for every $X \subset Y \subset X^{**}$ [30]. Moreover, $n(X) = n(X^*)$ holds true when X is a C^* -algebra [21]. Another interesting result is the following: if the dual of a Banach space X has the Radon Nikodym Property (RNP) and $n(X) = 1$, then $n(X^*) = 1$ as well (see [7] for details). It is still an open question whether $n(X) = n(X^*)$ holds true for every Banach space with the RNP.

Remark 3.14. *The polynomial numerical index of order k of a Banach space was first introduced in [10]. Let E and F be real or complex Banach spaces and $\mathcal{L}({}^k E : F)$ denote the Banach space of continuous, k -linear mappings $E^k := E \times \cdots \times E$ into F . $P : E \rightarrow F$ is called continuous k -homogeneous polynomial if there is $A \in \mathcal{L}({}^k E : F)$ such*

that $P(x) = A(x, \dots, x)$ for all $x \in E$. Let $\mathcal{P}({}^k E : F)$ denote the Banach space of continuous k -homogeneous polynomials of E into F , endowed with the polynomial norm $\|P\| = \sup_{x \in B_E} \|P(x)\|$.

For each $P \in \mathcal{P}({}^k E : F)$, the numerical range of P is the subset $W(P)$ of the scalar field defined by

$$W(P) = \{x^*(Px) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

and the numerical radius of P is given by

$$\nu(P) = \sup\{|\lambda| : \lambda \in W(P)\}.$$

The polynomial numerical index of order k of the space E , $n^{(k)}(E)$ is defined as:

$$n^{(k)}(E) = \inf\{\nu(P) : P \in S_{\mathcal{P}({}^k E : E)}\}$$

or equivalently it is the greatest constant c such that

$$c\|P\| \leq \nu(P)$$

for every $P \in \mathcal{P}({}^k E : E)$. Note that polynomial numerical index of order 1 is the ‘‘classical’’ numerical index and just like in the case of the classical numerical index, we have: $0 \leq n^{(k)}(E) \leq 1$ and $n^{(k)}(E) > 0$ if and only if $\|\cdot\|$ and $\nu(\cdot)$ are equivalent norms on $\mathcal{P}({}^k E : E)$. Clearly if E_1 and E_2 are isometrically isomorphic Banach spaces, then $n^{(k)}(E_1) = n^{(k)}(E_2)$ hold. For more on the properties of the polynomial numerical index of order k , we refer the reader to [16], [23], [10] and [22].

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