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DIAGONAL OPERATORS, S-NUMBERS AND BERNSTEIN PAIRS

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Abstract. *Replacing the nested sequence of "finite" dimensional subspaces by the nested sequence of "closed" subspaces in the classical Bernstein lethargy theorem, we obtain a version of this theorem for the space $\mathcal{B}(X,Y)$ of all bounded linear maps. Using this result and some properties of diagonal operators, we investigate conditions under which a suitable pair of Banach spaces form an exact Bernstein pair. We also show that many "classical" Banach spaces, including the couple $(L_p[0,1], L_q[0,1])$ form a Bernstein pair with respect to any sequence of s-numbers (s_n), for $1 < p < \infty$ and $1 \leq q < \infty$.*

1 Introduction

s-Numbers. Let X and Y be Banach spaces and $\mathcal{B}(X,Y)$ denote the space of all bounded linear maps from X into Y. According to A. Pietsch [10, 11], a map s which to each bounded linear map T from one Banach space to another such space assigns a unique sequence $(s_n(T))$ is called a s-function if for all Banach spaces W, X, Y, Z :
i) $||T|| = s_1(T) \geq s_2(T) \geq ... \geq 0$ for all $T \in \mathcal{B}(X,Y)$
ii) $s_n(S+T) \leq s_n(S) + ||T||$ for all $S, T \in \mathcal{B}(X,Y)$, and all $n \in \mathbb{N}$
iii) $s_n(RST) \leq ||R|| s_n(S) ||T||$ for all $T \in \mathcal{B}(X,Y), S \in \mathcal{B}(Y,Z)$ and $R \in \mathcal{B}(Z,W)$
iv) If $T \in \mathcal{B}(X,Y)$ and $rank(T) < n$, then $s_n(T) = 0$
v) $s_n(I) = 1$ for all $n \in \mathbb{N}$,
where I is the identity map of $l^2_n = \{x \in l_2 : x_i = 0 \text{ if } i > n\}$.
$s_n(T)$ is called *the n-th s-number* of the operator T.
Now we turn to some special s-numbers. Their definitions are:
-Approximation numbers :

$$a_n(T) := inf\{||T - S|| : rank(S) < n\} \text{ where } T, S \in \mathcal{B}(X,Y).$$

-Gelfand numbers:

$$c_n(T) := inf\{||TJ^X_M|| : codim(M) < n\},$$

where $T \in \mathcal{B}(X,Y)$ and J^X_M is the embedding from M into X.
- Kolmogorov numbers (or n-widths) :

$$d_n(T) := inf\{||Q^Y_N T|| : dim(N) < n\}$$

where $T \in \mathcal{B}(X,Y)$ and Q^Y_N is the canonical map from Y to Y/N.
For relations between several kind of s-numbers we refer to [10, 11].

Bernstein's "Lethargy" Theorem [1] *Let $V_1 \subset V_2 ....$ be a nested sequence of distinct finite dimensional vector subspaces of a Banach space X. Let (ε_n) be a decreasing sequence of* *nonnegative numbers tending to 0. Then there exist $x \in X$ such that $dist(x,V_n) = \varepsilon_n$ for $n = 1,2,\ldots$.*

Besides being a very important result of the constructive theory of functions, Bernstein's lethargy theorem can also be applied to the theory of quasi-analytic functions of several complex variables. For this and other applications of this theorem see [12] and [13] . (See also [6, 7, 8, 9] where the cases of F-spaces and Modular spaces are considered.)

The aim of this paper is to investigate the following

Problem. *Given Banach spaces X and Y and a sequence of s-numbers (s_n) is it true that for any decreasing sequence of nonnegative real numbers $\varepsilon_n \to 0$, there exist $T \in \mathcal{B}(X,Y)$ and a constant M depending only on T such that for every $n \in \mathbb{N}$*

$$\varepsilon_n \leq s_n(T) \leq M\varepsilon_n. \tag{1.1}$$

Two Banach spaces X and Y satisfying (1.1) will be called a Bernstein pair with respect to the sequence of s-numbers (s_n). We denote Bernstein pairs by (X,Y). If $M = 1$, then (X,Y) is called an exact Bernstein pair.

Our goal is to show that " classical" Banach spaces form Bernstein pairs, with respect to any sequence of s-numbers. This is quite a different approach from that of [4] in which Bernstein pairs are defined only with respect to approximation numbers. Moreover we replace classical l_p-spaces with more general sequence spaces. The main results of this paper are Theorems 2.9 and 3.2.

In the sequel the following notion will be needed.

Diagonal Operators. Let X and Y be Banach spaces. Let $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ be linearly independent sequences. An operator $D_\varepsilon \in \mathcal{B}(X,Y)$ with $D_\varepsilon x_n = \varepsilon_n y_n$ where (ε_n) is some fixed scalar sequence, is called a diagonal operator determined by (ε_n), with respect to (x_n) and (y_n). The set of all diagonal operators from X to Y will be denoted by $\mathcal{D}(X,Y)$.

2 Diagonal Operators and Approximation Numbers

Let X be a normed space and let V_n be a closed subspace of X. The set of all projections from X onto V_n will be denoted by $\mathcal{P}(X,V_n)$. We start with the following version of the Bernstein "Lethargy" theorem.

Theorem 2.1 *[8] Let $V_1 \subset V_2 \subset \ldots$ be a nested sequence of distinct closed subspaces of a Banach space X. Assume that $P_n \in \mathcal{P}(X,V_n)$ are so chosen that for every $n \in \mathbb{N}$, there exits $v_n \in V_{n+1} \setminus V_n$ such that*

$$P_i v_n = 0 \; for \; i = 1,2,3,\ldots,n. \tag{2.1}$$

Let (ε_n) be a decreasing sequence of nonnegative numbers tending to 0. Then, there exists an $x \in X$ with $\|x - P_n x\| = \varepsilon_n$ for $n = 1,2,\ldots$.

Corollary 2.2 *Let X, V_n, P_n and (ε_n) be as in Theorem 2.1. Suppose that there is $M > 0$ such that $\|I - P_n\| \leq M$ for $n = 1,2,\ldots$. Then there exists $x \in X$ such that*

$$\varepsilon_n / M \leq dist(x,V_n) \leq \varepsilon_n. \tag{2.2}$$

Proof. The proof is a simple consequence of Theorem 2.1 and the inequality

$$||x - P_n x|| \leq ||I - P_n|| dist(x, V_n).$$

□

Next we consider Banach spaces X, Y and the space $\mathcal{B}(Y,X)$ and state Theorem 2.1 for $\mathcal{B}(Y,X)$.

Proposition 2.3 *Let X, V_n, P_n and (ε_n) be as in Theorem 2.1. Then for every Banach space Y, there exists $L \in \mathcal{B}(Y,X)$ such that*

$$||L - W_n L|| = \varepsilon_n \text{ for } n = 1,2,..., \tag{2.3}$$

where $W_n \in \mathcal{P}(\mathcal{B}(Y,X), \mathcal{B}(Y,V_n))$ is defined by:

$$W_n L = P_n \circ L. \tag{2.4}$$

Proof. We need to show that for every $n \in N$, there exists $L_n \in \mathcal{B}(Y, V_{n+1}) \setminus \mathcal{B}(Y, V_n)$ such that $W_i L_n = 0$ for $i = 1,2,...,n$. To do this, take $v_n \in V_{n+1} \setminus V_n$ such that $P_i v_n = 0$ for $i = 1,2,...n$. Set $L_n y = f(y) v_n$, where $f \in Y^* \setminus \{0\}$. Then, for every $y \in Y$ and $i = 1,2,...,n$,

$$W_i(L_n y) = (P_i L_n) y = P_i(f(y) v_n) = f(y) P_i v_n = 0.$$

□

Notation 2.4 *For any $m \in \mathbb{N}$, set*

$$V_m = \{(x_n) \in \mathbb{K}^{\mathbb{N}} : x_n = 0 \text{ for } n > m\} \text{ and } V = \bigcup_{m=1}^{\infty} V_m. \tag{2.5}$$

Let $Y_o = (V, ||\cdot||_1)$ and $X_o = (V, ||\cdot||_2)$. Taking completions one can assume that X_o and Y_o are Banach spaces. Throughout this paper we will assume the following "order preserving" condition on the norm of X_o :

$$\text{If } |x_i| \leq |y_i| \text{ for } i = 1,2,..., \text{ then } ||x||_2 \leq ||y||_2 \text{ for all } x, y \in X_o; \tag{2.6}$$

and

$$||e_i||_1 = ||e_i||_2 = 1 \text{ for } i = 1,2,.... \tag{2.7}$$

Note that for a Banach space which satisfies (2.6), one also has

$$||P_n||_2 = ||I - P_n||_2 = 1 \text{ for any } n \in \mathbb{N}, \tag{2.8}$$

where P_n is a projection from X_o onto V_n defined by $P_n x = (x_1, x_2, ... x_n, 0, 0.....)$.

Corollary 2.5 *For any decreasing sequence (ε_n) of nonnegative numbers tending to zero, there exists an $L \in \mathcal{B}(Y_o, X_o)$ [$L \in \mathcal{D}(Y_o, X_o)$ respectively] such that*

$$dist(L, \mathcal{B}(Y_o, V_n)) = \varepsilon_n$$

$$[dist(L, \mathcal{D}(Y_o, V_n)] = \varepsilon_n], \text{ respectively for } n = 1,2,3....$$

Proof. Since the case of linear operators follows from (2.2), (2.4) and (2.8) and Corollary 2.2, we restrict ourselves to the case of diagonal operators. Define for $n \in \mathbb{N}$, and $L \in \mathcal{D}(Y_o, X_o)$, $W_n(L) = P_n \circ L$. It is clear that W_n is a projection from $\mathcal{D}(Y_o, X_o)$ onto $\mathcal{D}(Y_o, V_n)$. Moreover, by (2.8), $||I - W_n|| = 1$. Now for $n \in \mathbb{N}$, define $L_n \in \mathcal{D}(Y_o, V_{n+1}) \setminus \mathcal{D}(Y_o, V_n)$ by $L_n x = x_{n+1} e_{n+1}$. It is clear that $W_i(L_n) = 0$ for $i = 1, 2, ..., n$. Hence by Theorem 2.1, there is an $L \in \mathcal{D}(Y_o, X_o)$ such that

$$\varepsilon_n = ||L - W_n L|| = dist(L, \mathcal{D}(Y_o, V_n)).$$

□

Consider for $n \in \mathbb{N}$, $(n-1)$-dimensional subspace V_{n-1} and an arbitrary $L \in \mathcal{B}(Y_o, X_o)$, then one always has:

$$dist(L, \mathcal{D}(Y_o, V_{n-1})) \geq dist(L, \mathcal{B}(Y_o, V_{n-1})) \geq a_n(L). \tag{2.9}$$

Now for the following two propositions, we investigate conditions under which

$$a_n(L) = dist(L, \mathcal{D}(Y_o, V_{n-1})) \text{ holds for } n = 1, 2, ... \tag{2.10}$$

Proposition 2.6 *The equality (2.10) holds true when $X_o = Y_o$, for any $D_\varepsilon \in \mathcal{D}(X_o)$.*

Proof. For $n \in \mathbb{N}$, let D_n and I_n denote the operators D_ε and I restricted to V_n, then from (2.6) we have $||D_n^{-1}|| = \varepsilon_n^{-1}$. Note that

$$1 = a_n(I_n) = a_n(D_n^{-1} \circ D_n) \leq ||D_n^{-1}|| a_n(D_n) = \varepsilon_n^{-1} a_n(D_n).$$

Therefore

$$a_n(D_\varepsilon) \geq a_n(D_n) \geq \varepsilon_n = ||D_\varepsilon - W_{n-1}(D_\varepsilon)|| = dist(D_\varepsilon, \mathcal{D}(X_o, V_{n-1}))$$

which together with (2.9) gives the desired equality. □

We need the following lemma due to V. D. Milman [10, 11] to prove the Proposition 2.8.

Lemma 2.7 *Let V be any subspace of $l_\infty^{(m)}$ such that $codim(V) < n$. Then there exists $x \in V$, with $||x|| = 1$, such that*

$$card\{k : |x_k| = 1\} \geq m - n + 1.$$

Proposition 2.8 *The equality (2.10) holds true for any diagonal operator $D_\varepsilon \in \mathcal{D}(l_\infty, X_o)$ or $D_\varepsilon \in \mathcal{D}(c_o, X_o)$ if the norm in X_o is symmetric.*

Proof. We only need to verify $a_n(D_\varepsilon) \geq dist(D_\varepsilon, \mathcal{D}(l_\infty, V_{n-1}))$. First observe that for any $D_\varepsilon \in \mathcal{D}(l_\infty, X_o)$ such that $\varepsilon_1 \geq \varepsilon_2 \geq .... \geq 0$,

$$dist(D_\varepsilon, \mathcal{D}(l_\infty, V_{n-1})) = ||(0, ..., \varepsilon_n, \varepsilon_{n+1}, ...)||_2.$$

Let $A : l_\infty \to X_o$ be an arbitrary operator of rank $\leq n-1$, say $A = \sum_{j=1}^{n-1} f_j(\cdot) x_j$ where $f_j \in l_\infty^*$ and $x_j \in X_o$. Take $m \geq n$ and define

$$S_m = \{y \in l_\infty^{(m)} : f_j(y) = 0 \text{ for } j = 1, 2, ..., n-1\}.$$

Then S_m is a subspace of $l_\infty^{(m)}$ of codimension $\leq n-1$. By Lemma 2.7 there exists $y^o \in S_m$ with $||y^o|| = 1$ and indices $j(1) < j(2) < ... < j(m-n+1) \in \{1,2,...,m\}$ such that $| y^o_{j(i)} | = 1$ for $1 \leq i \leq m-n+1$. Extending y^o to l_∞ by setting $y^o_j = 0$ for $j > m$, we obtain

$$||D_\varepsilon - A|| \geq ||(D_\varepsilon - A)y_o||_2 = ||D_\varepsilon y_o||_2$$

$$= ||(\varepsilon_j y_{j(1)})_{j(1)}, ..., (\varepsilon_{j(m-n+1)} y_{j(m-n+1)})_{j(m-n+1)}, 0, ...)||_2.$$

By the ordering property (2.6) and symmetry of $||\cdot||_2$, the last term above is greater or equal to

$$||0,0,...,\varepsilon_{j(1)},\varepsilon_{j(2)},...,\varepsilon_{j(m-n+1)},0,...||_2 \geq ||0,0,...,0,\varepsilon_n,\varepsilon_{n+1},...,\varepsilon_{m-n+1},0,...||_2.$$

Since A and m were arbitrary, $a_n(D_\varepsilon) \geq ||(0,...,\varepsilon_n,\varepsilon_{n+1},....)||_2$. □

Note that Proposition (2.8) is a generalization of Theorem 1.8 of [4]. In Proposition (2.8) we replace l_p-spaces of [4] by arbitrary symmetric spaces. Also, in [5] or [10, p. 159] it is shown that, if $1 < q < p < \infty$ and $T : l_p \to l_q$ is a diagonal operator determined by (ε_n), then

$$a_n(T) = (\sum_{k=n}^{\infty} | \varepsilon_k |^r)^{1/r} \text{ where } r^{-1} + p^{-1} = q^{-1}. \tag{2.11}$$

By the Hölder inequality applied to r/q and p/q one can show that (2.10) also holds true in this case.

The following theorem gives a characterization of spaces which can form Bernstein pairs with respect to approximation numbers.

Theorem 2.9 *The following pairs of Banach spaces form exact Bernstein pairs with respect to the sequence of approximation numbers* (a_n).
a. (X_o, X_o), *where* X_o *is defined in Notation 2.4.*
b. (c_o, X_o) *and* (l_∞, X_o), *provided that the norm on* X_o *is symmetric.*
c. (X_o, l_1), *provided* X_o *is reflexive.*

Proof. a. follows from Cor. 2.5 and Prop. 2.6;
b. follows from Cor. 2.5 and Prop. 2.8.
To prove **c**, first note that if X_o satisfies condition (2.6) so does X_o^*; therefore, by **b.**, (c_o, X_o^*) is an exact Bernstein pair. Now using the well known fact [11, p. 239] that for any compact operator T from X to Y, $a_n(T) = a_n(T^*)$, we conclude that $(X_o, l_1) = (X_o^{**}, c_o^*)$ is a Bernstein pair. □

3 S-numbers and Bernstein Pairs

We start with a simple lemma, which proof will be omitted.

Lemma 3.1 *Let* X, Y, *and* Z *be Banach spaces with* $X \subset Z$. *Let* $T \in \mathcal{B}(X,Y)$ *and* P *be a projection from* Z *onto* X. *Then*

$$s_n(T) \leq s_n(T \circ P) \leq s_n(T)||P||.$$

The following theorem states the conditions needed on an arbitrary s-number in order (X_o, X_o) to become a Bernstein pair with respect to any sequence of s-numbers (s_n).

Theorem 3.2 *Suppose for a sequence of s-numbers (s_n), there exists $C > 0$ such that*

$$s_n(I : (V_n, \|\cdot\|_2) \to (V_n, \|\cdot\|_2)) \geq C$$

for every n, where V_n is a subspace of X_o defined as in (2.5). Then (X_o, X_o) is a Bernstein pair with respect to (s_n).

Proof. Fix a decreasing sequence (ε_n) of positive numbers tending to zero. Consider the diagonal operator $D_\varepsilon \in \mathcal{D}(X_o)$ constructed as in Proposition 2.6 satisfying $a_n(D_\varepsilon) = \varepsilon_n$ for $n = 1, 2, 3, ...$; D_n will denote the operator D_ε restricted to V_n. Then $\|D_n^{-1}\| = \varepsilon_n^{-1} = (a_n(D_\varepsilon))^{-1}$. Next observe that

$$C \leq s_n(I : V_n \to V_n) = s_n(D_n^{-1} \circ D_n) \leq \|D_n^{-1}\| s_n(D_n).$$

But by Lemma 3.1,

$$s_n(D_n) = s_n(D_\varepsilon \circ P_n) \leq s_n(D_\varepsilon)\|P_n\|.$$

Therefore $C \leq \varepsilon_n^{-1} s_n(D_\varepsilon)$. On the other hand $s_n(D_\varepsilon) \leq a_n(D_\varepsilon) = \varepsilon_n$. □

Notice that the condition stated on the s-numbers in Theorem 3.2 is not an artificial one. This condition is satisfied by Approximation, Gelfand and Kolmogorov numbers, as stated in the next corollary.

Corollary 3.3 *(X_o, X_o) is an exact Bernstein pair with respect to (d_n) and (c_n).*

Proof. (X_o, X_o) is an exact Bernstein pair with respect to (d_n) because $d_n(I : V_n \to V_n) = 1$. Since $c_n(T) = d_n(T^*)$ for any linear operator [11, p. 95], it is also an exact Bernstein pair with respect to (c_n). □

The next proposition will permit us to construct some examples of Bernstein pairs. We omit a routine proof.

Proposition 3.4 *Suppose (X, Y) is a Bernstein pair with respect to (s_n). Suppose that a Banach space W contains an isomorphic and a complementary copy of X, and a Banach space V contains an isomorphic copy of Y. Then (W, V) is a Bernstein pair with respect to (s_n).*

Corollary 3.5 *For $1 < p < \infty$ and $1 \leq q < \infty$, the couple $(L_p[0,1], L_q[0,1])$ form a Bernstein pair with respect to any sequence of s-numbers (s_n).*

Proof. The corollary follows from the fact that (l_2, l_2) is a Bernstein pair with respect to any sequence of s-numbers (s_n) [11] and the fact that for every p, $1 \leq p < \infty$, $L_p[0,1]$ contains a subspace isomorphic to l_2 and complemented in $L_p[0,1]$ for $p > 1$ [14, p. 85]. □

Corollary 3.6 **i)** *Let Y be a separable Banach space and assume that (X, Y) is a Bernstein pair with respect to (s_n). Then (X, l_∞) is a Bernstein pair with respect to (s_n).*
ii) *Let X^* be a separable Banach space, assume (X, Y) is a Bernstein pair with respect to (a_n). Then (X, c_o) and $(l_1, X^*)$are Bernstein pairs with respect to (a_n).*

Proof. i) follows from the fact that every separable Banach space Y is linearly isometric to a subspace of l_∞.
For **ii)** observe that (X, l_∞) is a Bernstein pair by **i),** then apply Lemma 4.10 of [4] to conclude that (X, c_o) is a Bernstein pair. If T is a compact operator, then $a_n(T) = a_n(T^*)$ for $n = 1, 2, \ldots$, will give the assertion for (l_1, X^*). □

Corollary 3.7 **i)** *Suppose (c_o, c_o) is a Bernstein pair with respect to (s_n). Then (X,Y) is a Bernstein pair with respect to (s_n), provided X and Y each contain an isomorphic copy of c_o, and X is separable.*
ii) *Suppose (l_1, l_1) is a Bernstein pair with respect to (s_n). Then (X,Y) is a Bernstein pair with respect to (s_n) provided X is a nonreflexive subspace of $L_1[0,1]$ containing a isomorphic copy of l_1 and Y containing a isomorphic copy of l_1.*

Proof. i) follows from Sobczyk's theorem [2, p. 71] which states that if a separable X contains an isomorphic copy of c_o, then X contains a complemented copy of c_o.
ii) follows from the Pełczyński- Kadeč theorem [2, p. 94], which states that if X is a nonreflexive subspace of $L_1[0,1]$, then X contains a subspace complemented in $L_1[0,1]$ and isomorphic to l_1. □

Corollary 3.8 *For $0 < q < p < 2$, $(l_p, L_q[\Omega, \mu])$ is a Bernstein pair with respect to (a_n), (c_n) and (d_n).*

Proof. It is known that [14, p. 94], if $0 < q < p < 2$ the real space $L_q[\Omega, \mu]$ contains a subspace isometric to l_p. Applying Theorem 3.2 to $X_o = l_p$, we see that $(l_p, L_q(\Omega, \mu))$ is a Bernstein pair with respect to the given s-numbers. □

Corollary 3.9 **i)** *If (l_∞, l_∞) is a Bernstein pair with respect to (s_n), and X and Y contain isomorphic copies of l_∞, then (X,Y) form a Bernstein pair with respect to (s_n).*
ii) *Let $L_f(\Omega, \Sigma, \mu), L_g(\Omega, \Sigma, \mu)$ be Orlicz spaces with nonatomic measure μ, where f, g do not satisfy Δ_2-condition if $\mu(\Omega)$ is infinite, and in case $\mu(\Omega)$ is finite, the Δ_2-condition at infinity is not satisfied. Here the norm on Orlicz spaces could be either Orlicz or Luxemburg norm. If (l_∞, l_∞) is a Bernstein pair with respect to (s_n), then $(L_f(\Omega, \Sigma, \mu), L_g(\Omega, \Sigma, \mu)$ form an exact Bernstein pair with respect to (s_n).*

Proof. i) By a theorem of Phillips [2, p. 21], if X and Y contain an isomorphic copy of l_∞, then X, Y contain one-complemented copies of l_∞.
ii) From [3, cor.2] we know that if f does not satisfy a suitable Δ_2-condition, then L_f has an isometric complemented copy of l_∞. □

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