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THE APPROXIMATION NUMBERS $\gamma_n(T)$ AND Q-PRECOMPACTNESS

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Abstract. We shall introduce the sequence $(\gamma_n(T))$ of approximation numbers for a linear operator T between p-Banach spaces $(0 < p \leqslant 1)$ and consider Q-precompactness in a metric linear space.

Introduction. Recently Pietsch [5] has introduced the concept of an approximation scheme on Banach spaces. The first author [1] has generalized this notion so that the usual approximation numbers can be deduced as special cases of an approximation schme, she then used these to generalize the concept of approximation numbers and Kolmogorov diameters in Banach spaces and studied the resulting concept of Q-compactness which is a proper extension of compactness.

In this paper we consider those results in the context of p-Banach spaces $(0 < p \leqslant 1)$ or general metric linear spaces. In Section 2 we introduce the sequence of numbers (γ_n) associated with a bounded set in a p-Banach space and study the relations between $\alpha_n(T)$, $\gamma_n(T)$ and $\delta_n(T)$ for an operator T between p-Banach spaces. In Section 3 we discuss the relation between $\alpha_n(T)$, $\gamma_n(T)$ and those of T' in Banach spaces. In the final section 4 we consider Q-precompactness in a metric linear space and prove a Dieudonné-Schwartz type characterization of Q-precompact sets in a p-normed space.

1. Preliminaries. Let E be a topological vector space over the field K of real or complex numbers and N be the set of all non-negative integers. For each $n \in N$, let $Q_n = Q_n(E)$ be a family of subsets of E satisfying the following conditions :

(1) $\{0\} = Q_0 \subset Q_1 \subset \cdots \subset Q_n \subset \cdots$,

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(2) $\lambda Q_n \subset Q_n$ for every $n \in N$ and $\lambda \in K$,

(3) $Q_n + Q_m \subset Q_{n+m}$ for every $n, m \in N$.

Then $Q(E) = (Q_n(E))_{n \in N}$ is called an *approximation scheme* on E. We shall simply use Q_n to denote $Q_n(E)$ if the context is clear.

Let U be a 0-neighborhood of E and D a bounded subset of E. Then the n-th *Kolmogorov diameter* $\delta_n(D, U; Q)$ of D with respect to U is defined by

$$\delta_n(D, U; Q) = \inf\{\lambda > 0 : D \subset \lambda U + A_n \text{ for some } A_n \in Q_n\}.$$

Let p be a number with $0 < p \leqslant 1$. Then, by a p-norm $\|\cdot\|$ on E we mean a map $\|\cdot\|$ of E into K which has the following properties :

(1) $\|x\| \geqslant 0$ for every $x \in E$, and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\lambda x\| = |\lambda| \|x\|$ for every $x \in E$ and $\lambda \in K$,

(3) $\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$ for every $x, y \in E$.

If a topology of E is induced by a p-norm, then E is said to be a p-normed space. If the p-normed space E is complete, then E is said to be a p-Banach space. We denote by U_E the closed unit ball of the p-Banach space E. Then U_E is an absolutely p-convex set, which means that $\lambda x + \eta y \in U_E$ for every $x, y \in U_E$ and every $\lambda, \eta \in K$ with $|\lambda|^p + |\eta|^p \leqslant 1$. Let E be a q-Banach space and F be a p-Banach space. We denote by $L(E, F)$ the vector space of all continuous linear mappings of E into F. Then, for each $T \in L(E, F)$ $\|T\| = \sup\{\|Tx\| : x \in U_E\}$ evidently defines a p-norm $\|\cdot\|$ on $L(E, F)$.

2. The numbers $\alpha_n(T)$, $\gamma_n(T)$ and $\delta_n(T)$. Let D be a bounded subset of a p-Banach space E. Now we defines the sequence $(\gamma_n(D, Q))$ by

$$\begin{aligned}\gamma_n(D) &\equiv \gamma_n(D, Q) \\ &= \inf\{\lambda > 0 : D \subset \lambda U_E + T(U_X) \text{ and } T(U_X) \subset A_n \text{ for some} \\ &\qquad p\text{-Banach space } X,\ T \in L(X, E) \text{ and } A_n \subset Q_n(E)\}.\end{aligned}$$

We start with the following proposition.

Proposition 1. *Let C and D be bounded subsets of a p-Banach space E. Then*

(1) $\gamma_n(\alpha D)=|\alpha|\gamma_n(D)$ for $n\in N$ and $\alpha\in K$,

(2) $\gamma_{m+n}(C+D)^p\leqslant\gamma_m(C)^p+\gamma_n(D)^p$ for $m,\ n\in N$.

The Kolmogorov diameter $\delta_n(D, U_E;Q)$ *has similar properties.*

Proof. (1) is trivial. To prove (2), let $\gamma_m(C)<\lambda$ and $\gamma_n(D)<\eta$. Then there exist p-Banach spaces X_i, $T_i\in L(X_i, E)$ $(i=1,2)$, $A_m\in Q_m$ and $A_n\in Q_n$ such that

$$C\subset\lambda U_E+T_1(U_1),\quad T_1(U_1)\subset A_m$$

and

$$D\subset\eta U_E+T_2(U_2),\quad T_2(U_2)\subset A_n,$$

where U_i denotes the unit ball of X_i $(i=1,2)$. Hence

$$C+D\subset(\lambda^p+\eta^p)^{1/p}U_E+T_1(U_1)+T_2(U_2)$$

and $T_1(U_1)+T_2(U_2)\subset A_m+A_n\in Q_{m+n}$. Let $X=X_1\times X_2$ be the set of all pairs (x_1, x_2) with $x_1\in X_1$ and $x_2\in X_2$. Then it is easily seen that X becomes a p-Banach space with respect to the maximum norm $\|(x_1, x_2)\|\equiv\max(\|x_1\|, \|x_2\|)$. Let P_i be the projection of X onto X_i. Then, by putting $T=T_1P_1+T_2P_2$, we obtain

$$T(U_X)=T_1P_1(U_X)+T_2P_2(U_X)=T_1(U_1)+T_2(U_2)\subset\widetilde{A}_{m+n}\in Q_{m+n}$$

and $C+D\subset(\lambda^p+\eta^p)^{1/p}U_E+T(U_X)$. Thus we have $\gamma_{m+n}(C+D)^p\leqslant\lambda^p+\eta^p$, and the assertion is proved for (γ_n). We can easily prove the assertion for $\delta_n(D, U_E;Q)$ and omit the proof.

We denote $\delta_n(D, U_E;Q)$ by $\delta_n(D)$.

Theorem 1. *Let D be a bounded subset of a p-Banach space E. If every element A_n of Q_n $(n\in N)$ is closed and absolutely p-convex, then $\gamma_n(D)=\delta_n(D)$.*

Proof. The inequality $\delta_n(D)\leqslant\gamma_n(D)$ is trivial. Assume that $\delta_n(D)<\lambda$. Then $D\subset\lambda U_E+A_n$ for some $A_n\in Q_n$. By assumption of boundedness of D, there exists $M>0$ such that $\|d\|\leqslant M$ for all $d\in D$. Since each element d of D is written as

$$d=\lambda u+a \quad (u\in U_E,\ a\in A_n),$$

we have

$$\|a\|^p=\|d-\lambda u\|^p\leqslant M^p+\lambda^p.$$

Hence $\|a\|\leqslant (M^p+\lambda^p)^{1/p}$. If we put $K=\max\{1,(M^p+\lambda^p)^{1/p}\}$ and $V=U_E\cap A_n$, then we have

$$D\subset \lambda U_E+KV,\quad KV\subset KA_n\in Q_n.$$

Let X be the vector space generated by A_n, q_V the Minkowski functional of V and $V_q=\{x\in E: q_V(x)\leqslant 1\}$. Then $V=V_q$ because V is closed. Let I be the identity map of the p-normed space (X,q_V) into E and $\widetilde{I}$ be the unique extension of I to the completion $\widetilde{X}$ of X. Then we obtain

$$D\subset \lambda U_E+K\widetilde{I}(V_q)\ \text{ and }\ K\widetilde{I}(V_q)\subset KA_n\in Q_n,$$

which implies $\gamma_n(D)\leqslant\lambda$, i.e., $\gamma_n(D)\leqslant\delta_n(D)$.

From now on we deal with the approximation numbers of operators of a q-Banach space E into a p-Banach space F. We now additionally assume that the given approximation scheme $Q=(Q_n)$ satisfies the following condition :

If $T\in L(E,F)$ and $A\subset A_n\in Q_n(E)$, then there exists $B_n\in Q_n(F)$ such that $T(A)\subset B_n$.

Given an approximation scheme Q, a q-Banach space E and a p-Banach space F, for $T\in L(E,F)$ the n-th *approximation number* $\alpha_n(T)$ with respect to Q is defined by

$$\alpha_n(T)=\inf\{\|T-S\|: S(E)\subset A_n \text{ for some } S\in L(E,F) \text{ and some } A_n\in Q_n(F)\},$$

where $S(E)$ means the image of E by S. We define $\delta_n(T)$ and $\gamma_n(T)$ by $\delta_n(T)=\delta_n(T(U_E))$ and $\gamma_n(T)=\gamma_n(T(U_E))$, respectively.

Theorem 2. *Let E be a q-Banach space, F a p-Banach space, G a r-Banach space, S, $T\in L(E,F)$ and $R\in L(F,G)$. Then*

(1) $\|T\|=\gamma_0(T)\geqslant\gamma_1(T)\geqslant\cdots\geqslant\gamma_n(T)\geqslant 0,$

(2) $\gamma_n(\lambda T) = |\lambda|\gamma_n(T)$ *for* $\lambda \in K$ *and* $n \in N$,

(3) $\gamma_{m+n}(S+T)^p \leqslant \gamma_m(S)^p + \gamma_n(T)^p$ *for* $m, n \in N$,

(4) *If* $p = r$, $\gamma_{m+n}(RT) \leqslant \gamma_m(R)\gamma_n(T)$ *for* $m, n \in N$.

These relations are also valid for $\alpha_n(T)$ *and* $\delta_n(T)$ *and the assertion* (4) *is valid for these numbers without the assumption* $p = r$.

Proof. Case of $\gamma_n(T)$. (1) is evident because $Q_0 = \{0\}$. (2) and (3) follow immediately from Proposition 1. To prove (4), let $\gamma_m(R) < \mu$ and $\gamma_n(T) < \lambda$. Then there exist p-Banach spaces $X, Y, A_m \in Q_m(G)$ and $B_n \in Q_n(F)$ such that

$$R(U_F) \subset \mu U_G + K(U_X), \quad K(U_X) \subset A_m,$$

$$T(U_E) \subset \lambda U_F + L(U_Y), \quad L(U_Y) \subset B_n$$

for suitable $K \in L(X, G)$ and $L \in L(Y, F)$. Then we obtain

$$RT(U_E) \subset \lambda R(U_F) + RL(U_Y)$$

$$\subset \lambda\mu U_G + \lambda K(U_X) + RL(U_Y).$$

Now, as in the proof of Proposition 1 (2), we can construct a p-Banach space $Z = X \times Y$ and an operator $M \in L(Z, G)$ such that $RT(U_E) \subset \lambda\mu U_G + M(U_Z)$ and $M(U_Z) \subset C_{m+n}$ for some C_{m+n} in $Q_{m+n}(G)$. Thus we obtain $\gamma_{m+n}(RT) \leqslant \lambda\mu$, which implies $\gamma_{m+n}(RT) \leqslant \gamma_m(R)\gamma_n(T)$.

Case of $\alpha_n(T)$. Let $\alpha_m(R) < \mu$ and $\alpha_n(T) < \lambda$. Then there exist $A_n \in Q_n(F)$ and $B_m \in Q_m(G)$ such that

$$\|T - L\| < \lambda, \quad L(E) \subset A_n,$$

$$\|R - K\| < \mu, \quad K(F) \subset B_m$$

for some $L \in L(E, F)$ and $K \in L(F, G)$. By letting $M = KT + (R-K)L$, we have $RT - M = (R-K)(T-L)$ and $M(E) = KT(E) + (R-K)L(E)$. Since $L(E) \subset A_n$, there exists $A'_n \in Q_n(G)$ such that $(R-K)L(E) \subset A'_n$ from the assumption on Q_n. On the other hand, $KT(E) \subset K(F) \subset B_m$. Hence we have $M(E) \subset C_{m+n}$ for $C_{m+n} = A'_n + B_m \in Q_{m+n}(G)$ and

$$\alpha_{m+n}(RT) \leqslant \|RT - M\| \leqslant \|R - K\| \|T - L\| < \mu\lambda,$$

which implies $\alpha_{m+n}(RT) \leqslant \alpha_m(R)\alpha_n(T)$.

Since the other inequalities for $\alpha_n(T)$ and $\delta_n(T)$ are easily verified, we omit the proof.

Let E and F be p-Banach spaces and $T \in L(E, F)$. T is said to be a *metric surjection* if T maps the open unit ball of E onto the open unit ball of F. A p-Banach space E is said to have the *metric lifting property* if for every metric surjection $Q \in L(F_0, F)$ and every $T \in L(E, F)$, given $\epsilon > 0$, we can find $S \in L(E, F_0)$ with $T = QS$ and $\|S\| \leqslant (1 + \epsilon) \|T\|$, where F and F_0 are arbitrary p-Banach spaces. The p-Banach space ℓ_I^p has the metric lifting property for any index set I (see [3]).

For a p-Banach space E we put $E^{\text{sur}} = \ell_I^p$, where $I = U_{E'}$ and we define a surjection Q_E of E^{sur} onto E by $Q_E(\xi_x) = \Sigma \xi_x x$ for $(\xi_x)_{x \in I} \in \ell_I^p$. Then Q_E is a metric surjection (see [3]).

Proposition 2. *Let E be a q-Banach space with the metric lifting property, F be a p-Banach space and $T \in L(E, F)$. Then the equality $\alpha_n(T) = \alpha_n(TQ_E)$ holds.*

Proof. Given $\epsilon > 0$, for the identity map I_E of E there exists $S \in L(E, E^{\text{sur}})$ such that $Q_E S = I_E$ and $\|S\| \leqslant (1 + \epsilon) \|I_E\| = 1 + \epsilon$. Hence

$$\alpha_n(TQ_E) \leqslant \alpha_n(T) = \alpha_n(TQ_E S) \leqslant \alpha_n(TQ_E) \|S\| \leqslant (1+\epsilon)\alpha_n(TQ_E),$$

which proves the proposition.

Let T be an operator of a q-Banach space E into a p-Banach space F and N be a closed vector subspace of F. We denote by Q_N^F the canonical surjection of F onto the quotient space E/N. We consider all closed subspaces N of F such that $N \subset A_n$ for some $A_n \in Q_n(F)$, and put $\beta_n(T) = \inf \|Q_N^F T\|$. Then we obtain a result similar to the known one on Kolmogorov numbers.

Theorem 3. *Let $T \in L(E, F)$. Then*

$$\alpha_n(TQ_E) = \beta_n(T) = \delta_n(T).$$

Proof. Since the equality $\beta_n(T) = \delta_n(T)$ follows immediately from the definition, we prove $\alpha_n(TQ_E) = \beta_n(T)$. Let N be a closed subspace of F such that $N \subset A_n$ for some $A_n \in Q_n(F)$ and let $\epsilon > 0$. Since E^{sur} has the metric lifting property, for the operator $Q_N^F TQ_E$ there exists $S \in L(E^{sur}, F)$ such that $Q_N^F S = Q_N^F TQ_E$ and $\|S\| \leqslant (1+\epsilon)\|Q_N^F T\|$. By setting $L = TQ_E - S$, we have an operator $L \in L(E^{sur}, F)$ such that $L(E^{sur}) \subset N \subset A_n$. Hence

$$\alpha_n(TQ_E) \leqslant \|TQ_E - L\| = \|S\| \leqslant (1+\epsilon)\|Q_N^F T\|,$$

which proves that $\alpha_n(TQ_E) \leqslant \beta_n(T)$.

To show the equality, given $\epsilon > 0$, we choose $L \in L(E^{sur}, F)$ such that $L(E^{sur}) \subset A_n$ for some A_n in $Q_n(F)$ and $\|TQ_E - L\| \leqslant (1+\epsilon)\alpha_n(TQ_E)$. If we put $N = L(E^{sur})$, then $N \subset A_n$ and

$$\|Q_N^F T\| = \|Q_N^F TQ_E\| = \|Q_N^F(TQ_E - L)\|$$
$$\leqslant \|TQ_E - L\| \leqslant (1+\epsilon)\alpha_n(TQ_E).$$

Therefore $\beta_n(T) \leqslant \alpha_n(TQ_E)$. Thus the theorem is proved.

3. **Relations between $\alpha_n(T)$, $\gamma_n(T)$ and those of T'.** In this section, for an operator T between Banach spaces we investigate the relations between the number $\alpha_n(T)$, $\gamma_n(T)$ and those of dual operators T'. For that purpose, we assume that each element $A_n \in Q_n$ is a closed subspace of a Banach space considered. We essentially follow the method of Astala [2].

First we need the following lemma according to Astala [2].

Lemma. *Let E, F, X_1 and X_2 be Banach spaces. Assume that E has the metric lifting property and operators $T \in L(E, F)$, $S \in L(X_1, F)$ and $R \in L(X_2, F)$ satisfy the condition.*

$$T(U_E) \subset S(U_{X_1}) + R(U_{X_2}).$$

Then for any $\epsilon > 0$, there are operators $K_i \in L(E, X_i)$ $(i = 1, 2)$ such that $T = SK_1 + RK_2$ and $\|K_i\| < 1 + \epsilon$.

Let $T \in L(E, F)$ and Q_E be the canonical surjection of E^{sur} onto E. Then, by Theorem 3 we have

$$\delta_n(T)=\alpha_n(TQ_E)\leqslant\alpha_n(T)\,\|TQ_E\|=\alpha_n(T),$$

i.e. $\delta_n(T)\leqslant\alpha_n(T)$, and so, by Theorem 1 we obtain $\gamma_n(T)\leqslant\alpha_n(T)$. We don't know whether the equality $\gamma_n(T)=\alpha_n(T)$ generally holds or not. The next result will be proved in this context.

Theorem 4. *Let E and F be Banach spaces and $T\in L(E, F)$. If E has the metric lifting property, then $\gamma_n(T)=\alpha_n(T)$.*

Proof. We must show the inequality $\alpha_n(T)\leqslant\gamma_n(T)$ holds. Let $\gamma_n(T)<\lambda$. Then there exist a Banach space X and $T\in L(X,F)$ such that

$$T(U_E)\subset\lambda U_F+L(U_X)\text{ and }L(U_X)\subset A_n$$

for some $A_n\in Q_n(F)$. By Lemma, for every ϵ with $0<\epsilon<1$, there are $K_1\in L(E,X)$ and $K_2\in L(E,F)$ such that $\|K_i\|\leqslant 1+\epsilon$ $(i=1,2)$ and

$$T=LK_1+\lambda I_FK_2=LK_1+\lambda K_2.$$

Then $\|T-LK_1\|=\lambda\|K_2\|\leqslant\lambda(1+\epsilon)$. On the other hand, since $LK_1(U_E)\subset L(2U_X)\subset 2A_n$, we obtain

$$LK_1(E)=LK_1\Big(\bigcup_{\mu>0}\mu U_E\Big)=\cup\mu LK_1(U_E)\subset\cup 2\mu A_n=A_n.$$

Hence $LK_1(E)\subset A_n$. Thus we have $\alpha_n(T)\leqslant\lambda$ and $\alpha_n(T)\leqslant\gamma_n(T)$, which proves the theorem.

Corollary. *Let $T\in L(E,F)$. Then*

$$\gamma_n(T)=\alpha_n(TQ_E)\leqslant\alpha_n(T).$$

Proof. From Theorem 4 and the fact that Q_E is a metric surjection, it follows that

$$\gamma_n(T)=\gamma_n(TQ_E)=\alpha_n(TQ_E)\leqslant\alpha_n(T)\|Q_E\|=\alpha_n(T).$$

We next study the relations between $\gamma_n(T)$, $\gamma_n(T')$ and $\gamma_n(T'')$. For $T\in L(E, F)$ we need the following symmetric property on an approximation scheme (Q_n):

If $T(A) \subset A_n \in Q_n(F)$ $(A \subset E)$, *then there exist a subset* $B \subset F'$ *and* $B_n \in Q_n(E')$ *such that* $T'(B) \subset B_n$.

Propostition 3. *Let* E *and* F *be Banach spaces*, $T \in L(E, F)$ *and* J_F *be the canonical injection of* F *into* F''. *Then* $\gamma_n(J_F T) = \gamma_n(T'')$. *In particular, if* F *is a dual space, then* $\gamma_n(T) = \gamma_n(T'')$.

Proof. For $x \in F$, $y \in F'$ and $z \in F''$, we have

$$\langle x, y\rangle = \langle y, J_F x\rangle = \langle J_F x, J_{F'} y\rangle = \langle x, (J_F)' J_{F'} y\rangle$$

and

$$\langle y, z\rangle = \langle (J_F)' J_{F'} y, z\rangle = \langle J_{F'} y, (J_F)'' z\rangle = \langle y, (J_{F'})' (J_F)'' z\rangle.$$

Hence we obtain

$$(J_F)' F_{F'} = I_{F'} \text{ and } (J_{F'})' (J_F)'' = I_{F''}. \qquad (*)$$

Since $\alpha_n(T') \leqslant \|T' - L'\| = \|T - L\|$, $\alpha_n(T') \leqslant \alpha_n(T)$ for every $T \in L(E, F)$. From Corollary of Theorem 4, we obtain

$$\gamma_n(T'') = \gamma_n(T'' Q_E'') \leqslant \alpha_n(T'' Q_E'') \leqslant \alpha_n(T Q_E) = \gamma_n(T''),$$

which proves the first assertion.

Assume that F is a dual space of a Banach space G. Then we apply the former formula in $(*)$ with F replaced by G to get

$$\gamma_n(T) = \gamma_n(J_G' J_F T) \leqslant \gamma_n(J_F T) \leqslant \gamma_n(T),$$

which completes the proof.

Let E and F be Banach spaces and $T \in L(E, F)$. Then T is said to be a *metric injection* if $\|Tx\| = \|x\|$ for all $x \in E$, and E is said to have the *metric extension property* if for every metric injection $J \in L(E_0, E)$ and every $T_0 \in L(E_0, F)$ we can find $T \in L(E, F)$ with $T_0 = TJ$ and $\|T\| = \|T_0\|$, where E and E_0 are Banach spaces. It is well known that the Banach space ℓ_I^∞ has the metric extension property for any index set I.

Theorem 5. *Let* E *and* F *be Banach spaces and* $T \in L(E, F)$. *If* F *has the* *metric extension property, then*

$$\gamma_n(T') = \alpha_n(T') = \alpha_n(T).$$

Proof. The inequalities $\gamma_n(T') \leqslant \alpha_n(T') \leqslant \alpha_n(T)$ follow from Corollary of Theorem 4 and the proof of Proposition 3. To verify the equalities, assume that $\gamma_n(T') < \lambda$. By Corollary of Theorem 4, there exists an operator L of $(F')^{sur}$ into E' such that

$$\| T' Q^s_{F'} - L \| < \lambda, \qquad (**)$$

where $Q^s_{F'}$ denotes the canonical surjection of $(F')^{sur}$ onto F. Since $Q^s_{F'}$ maps the unit ball of $(F')^{sur}$ onto the unit ball of F', $(Q^s_{F'})'$ is a metric injection of F'' into F_0, where F_0 denotes the dual of the Banach space $(F')^{sur}$. By letting $J = (Q^s_{F'})' J_F$ we obtain a metric injection J of F into F_0. As F has the metric extension property, there exists $P \in L(F_0, F)$ such that $PJ = I_F$ and $\|P\| = \|I_F\| = 1$ for the identity map I_F of F. Now the inequality $(**)$ implies

$$\begin{aligned}\alpha_n(T) &\leqslant \|T - PK'J_E\| = \|P(JT - K'J_E)\| \leqslant \|JT - K'J_E\| \\ &= \|(Q^s_{F'})' J_F - K'J_E\| = \|(Q^s_{F'})' T'' J_E - K'J_E\| \\ &\leqslant \|(Q^s_{F'})' T'' - K'\| \leqslant \|T' Q^s_{F'} - K\| < \lambda.\end{aligned}$$

Therefore $\alpha_n(T) \leqslant \gamma_n(T')$ and we complete the proof.

Corollary. *Let $T \in L(E, F)$. If E has the lifting property and F has the extension property, then*

$$\gamma_n(T) = \gamma_n(T') \quad \text{and} \quad \delta_n(T) = \delta_n(T').$$

Proof. The assertion is clear from Theorems 1, 4 and 5.

4. Q-precompactness of bounded sets in a metrizable topological vector space.

Let E be a metrizable topological vector space. A bounded subset D of E is said to be *Q-precompact* if $\delta_n(D, U; Q) \to 0 \, (n \to \infty)$ for each 0-neighbourhood U of E. We assume that each $A_n \in Q_n \, (n \in N)$ is separable. Then it is evident that Q-precompact sets are separable. A sequence $(x_{n,k})_{n,k \in N}$ in E is said to be of *order* (c_0) if the following hold :

(1) For every $n \in N$ there exists an $A_n \in Q_n$ and $(x_{n,k})_k \subset A_n$,

(2) $x_{n,k} \to 0$ as $n \to \infty$ uniformly in k.

Theorem. *Suppose that E is a metrizable topological vector space with an approximation scheme $\{Q_n\}$ and each element $A_n \in Q_n$ is solid (i.e. $|\lambda| A_n \subset A_n$ for $|\lambda| \leq 1$) and let p be an arbitrary positive number. If a bounded subset D of E is Q-precompact, then there exists an order (c_0) sequence $(x_{n,k})_k$ such that*

$$D \subset \{ \sum_{n \in N} \alpha_n x_{n,k(n)} : x_{n,k(n)} \in (x_{n,k})_k, \sum_{n \in N} |\alpha_n|^p \leq 1 \}.$$

In particular, if E is a p-normed space $(0 < p \leq 1)$, then the converse holds.

Proof. Suppose that D is Q-precompact. Let (U_n) $(n \in N)$ be a base of 0-neighborhoods of E such that $U_{n+1} + U_{n+1} \subset U_n$ and the diameter of U_n converges to 0 as $n \to \infty$, and let α be a scalar with $|\alpha| \geq 1$. Then there exist an integer $n_0 \in N$ and $A_{n_0} \in Q_n$ such that

$$\alpha D \subset U_1 + A_{n_0} \tag{1}$$

because the set αD is Q-precompact. Let F_{n_0} be a countable dense subset of A_{n_0}. Then we have $(\alpha D + U_0) \cap F_{n_0} \neq \phi$. In fact, for every $d \in D$ there exist $u \in U_1$ and $a \in A_{n_0}$ with $\alpha d = u + a$ by (1). Since F_{n_0} is dense in A_{n_0}, there exist $u' \in U_1$ and $w \in F_{n_0}$ with $a = u' + w$. Therefore $\alpha d = u + a = (u + u') + w \in U_1 + U_1 + F_{n_0} \subset U_0 + F_{n_0}$, i.e. $\alpha D \subset U_0 + F_{n_0}$. Thus $(\alpha D + U_0) \cap F_{n_0} \neq \phi$. We put

$$B_0 = (\alpha D + U_0) \cap F_{n_0}. \tag{2}$$

Then for every $d \in D$ there exist $u \in U_0$ and $w \in F_{n_0}$ with $\alpha d = u + w$, and so $w \in B_0$ and then

$$\alpha D \subset B_0 + U_0.$$

We put

$$D_0 = (\alpha D - B_0) \cap U_0. \tag{3}$$

Then D_0 is Q-precompact. In fact, for every $\epsilon > 0$ and 0-neighborhood U_k there exist $m \in N$ and $A_m \in Q_m$ with $\alpha D \subset \epsilon U_k + A_m$. From $B_0 \subset F_{n_0} \subset A_{n_0}$ it follows that

$$D_0 \subset \alpha D - B_0 \subset eU_k + (A_m + A_{n_0})$$

and $A_m + A_{n_0} \in Q_{m+n_0}$. Therefore D_0 is Q-precompact. Now by (3), for every $d \in D$ we can find $b_0 \in B_0\ (\subset F_0)$ such that

$$\alpha d - b_0 \in D_0 \ \ (\subset U_0). \tag{4}$$

If we put $(x_{0,k})_k = \{b_0 \in B_0 : d \in D\}$, then $(x_{0,k})_k$ is at most countable because $b_0 \in F_{n_0}$.

Since αD_0 is Q-precompact, as before, there exist $n_1 \in N$ and $A_{n_1} \in Q_{n_1}$ such that $\alpha D_0 \subset U_2 + A_{n_1}$, then we have $(\alpha D_0 + U_1) \cap F_{n_1} \neq \phi$ for every countable dense set F_{n_1} of A_{n_1}. We put

$$B_1 = (\alpha D_0 + U_1) \cap F_{n_1} \text{ and } D_1 = (\alpha D_0 - B_1) \cap U_1. \tag{5}$$

Then for every $d_0 \in D_0$ we can find $b_1 \in B_1\ (\subset F_{n_1})$ such that $\alpha d_0 - b_1 \in D_1$ $(\subset U_1)$, i.e. $\alpha(\alpha d - b_0) - b_1 = \alpha^2 d - (\alpha b_0 + b_1) \in D_1$. Then the set $(x_{1,k})_k = \{b_1 \in B_1 : d_0 \in D_0\}$ is at most countable. By continuing the same process we can define

$$B_m = (\alpha D_{m-1} + U_m) \cap F_{n_m} \tag{6}$$

for every countable dense set F_{n_m} of A_{n_m}. Then

$$\alpha D_{m-1} \subset B_m + U_m,$$

and we put

$$D_m = (\alpha D_{m-1} - B_m) \cap U_m. \tag{7}$$

Suppose that for every $d \in D$ there exist $b_i \in B_i$ $(0 \leq i \leq m-1)$ such that $d' = \alpha^{m-1} d - (\alpha^{m-2} b_1 + \ldots + b_{m-1}) \subset D_{m-1}$. Since for every $d_{m-1} \in D_{m-1}$ there exists $b_m \in B_m$ with $\alpha d_{m-1} - b_m \in D_m$, we can find $b_m \in B_m$ such that $\alpha d' - b_m \in D_m$, i.e.

$$\alpha^m d - (\alpha^{m-1} b_1 + \ldots + b_m) \in D_m.$$

Therefore $d - (\frac{1}{\alpha} b_0 + \ldots + \frac{1}{\alpha^m} b_m) \in \frac{1}{\alpha^m} D_m \subset \frac{1}{\alpha^m} U_m \subset U_m$, i.e. $d =$

$\sum_{n\in N} \frac{1}{\alpha^{n+1}} b_n$. Now we put $(x_{n,k})_k = \{b_n \in B_n : \alpha D_{n-1} - b_n \in D_n\}$. Then for every $k \in N$, $x_{n,k} \in B_n \subset \alpha D_{n-1} + U_n \subset \alpha U_{n-1} + U_{n-1} \subset \alpha(U_{n-1} + U_{n-1}) \subset \alpha U_{n-2}$. Therefore $(x_{n,k})_k \to 0$ as $n \to \infty$ uniformly with respect to k. If we put $\alpha = 2^{1/p}$ and $\lambda_n = 1/\alpha^{n+1}$, then we have $d = \sum_{n\in N} \lambda_n b_n$, and $\sum_{n\in N} |\lambda_n|^p \leqslant 1$.

Now we have shown that every element x of D is written in the following form

$$x = \sum_{i\in N} \alpha_{n_i} x_{n_i,k(n_i)}, \quad \sum_{i\in N} |\alpha_{n_i}|^p \leqslant 1,$$

where $x_{n_i,k(n_i)} \in F_{n_i} \subset A_{n_i}$ and F_{n_i} is dense in $A_{n_i} \in Q_{n_i}$. For each m with $n_i < m < n_{i+1}$ we put $\alpha_m = 0$ and $(x_{m,k})_k = (x_{n_i,k})_k$. Then $(x_{m,k})_k \subset A_{n_i} \in Q_{n_i} \subset Q_m$ and we have

$$x = \sum_{i\in N} \alpha_{n_i} x_{n_i,k(n_i)} = \sum_{m\in N} \alpha_m x_{m,k(m)}$$

and $\sum_{m\in N} |\alpha_m|^p = \sum_{n\in N} |\alpha_{n_i}|^p \leqslant 1$.

Thus the assertion is proved.

Conversely, suppose that E is a p-normed space and for each $n \in N$ there exist $A_n \in Q_n$ and $(x_{n,k})_k \subset A_n$ such that $x_{n,k} \to 0$ as $n \to \infty$ uniformly in k and

$$D \subset \{\sum_{n\in N} \alpha_n x_{n,k(n)} : x_{n,k(n)} \in (x_{n,k})_k, \sum_{n\in N} |\alpha_n|^p \leqslant 1\} \equiv C.$$

Thus it is enough to show that C is Q-precompact. By the assumption, for every $\epsilon > 0$ there exists $m \in N$ such that $\|x_{n,k}\| < \epsilon$ for all k and $n \geqslant m$. Since each element $c \in C$ can be written as

$$c = \sum_{n=0}^{m} \alpha_n x_{n,k(n)} + \sum_{n=m+1}^{\infty} \alpha_n x_{n,k(n)},$$

we have $\sum_{n=0}^{m} \alpha_n x_{n,k(n)} \in \sum_{n=0}^{m} \alpha_n A_n \subset \sum_{n=0}^{m} A_n \equiv \tilde{A}_{m^2} \in Q_{m^2}$ and for $\nu = \sum_{n=m+1}^{\infty} \alpha_n x_{n,k(n)}$ we have

$$\|\nu\|^p \leqslant \sum_{n=m+1}^{\infty} |\alpha_n|^p \|x_{n,k(n)}\|^p \leqslant \epsilon^p (\sum_{n=m+1}^{\infty} |\alpha_n|^p) \leqslant \epsilon^p,$$

i.e. $v \in \epsilon U_E$. Thus $C \subset \epsilon U_E + \widetilde{A}_{m^2}$ and so $\delta_n(C) \to 0$ as $n \to \infty$. Therefore D is Q-precompact.

Remark. In case that E is a p-normed space and Q_n is the set of all at most n-dimensional subspaces of E, we can show that the Q-precompactness of a bounded subset D coincides with the usual definition of the precompactness of D. In other words, $\delta_n(D) \to 0$ as $n \to \infty$ if and only if, given $\epsilon > 0$, there exists a finite subset $\{x_1, x_2, \dots, x_n\} \subset D$ such that $D \subset \bigcup_{i=1}^{n} \{x_i + \epsilon U_E\}$.

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