

In this lecture, we will see some connections between my area of research, a subfield of low-dimensional topology known as *knot theory*, and linear algebra. We'll start with some preliminaries about knots, then see two types of knot invariants defined using linear algebra: the *Alexander Polynomial* and *Reshetikhin-Turaev Invariants* also known as *Quantum Invariants*.

Knots

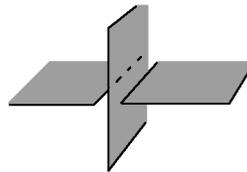
A *knot* is a simple closed curve in \mathbb{R}^3 . Here *simple* means “does not intersect itself” and *closed* means “does not have loose endpoints”. Knots are usually considered as *topological* objects, meaning that if a knot K can be continuously deformed into another knot K' , then K and K' are considered to be the same knot. A continuous deformation is more formally called an *ambient isotopy*; it is a continuous function $F : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$ such that $F(\vec{x}, t)$ is one-to-one for every $t \in [0, 1]$ with $F(S^1, 0) = K$ and $F(S^1, 1) = K'$ where S^1 is the unit circle. By thinking of t as a time variable, an ambient isotopy can be pictured as a movie changing one knot to the other in a continuous way, i.e. without cutting or moving the knot through itself.

The modern mathematical study of knots was started by Lord Kelvin in the late 19th century, who thought that atoms might be knots in the luminiferous ether, the hypothetical space-filling substance thought to carry light waves. Kelvin’s idea was to try to find the various possible different types of knots and match them up with the elements in the periodic table; in fact, he was trying to do chemistry, not topology! Sadly, the luminiferous ether was shown not to exist by Michelson-Morley, so Kelvin’s idea was wrong; nevertheless, Kelvin’s efforts got mathematicians (including Gauss himself) interested in the idea of classifying knots.

To prove that two knots K and K' are ambient isotopic, one approach would be to find parametrizations of K and K' and find a formula for an ambient isotopy. However, this is generally more work than is necessary. Since knots are considered as *topological* objects, we don’t really need an exact parametrization; instead, we can represent knots with pictures known as *knot diagrams*; these are projections or shadows of the knot on a plane, with understrands drawn “broken” so we can tell which strand passes over and which under.

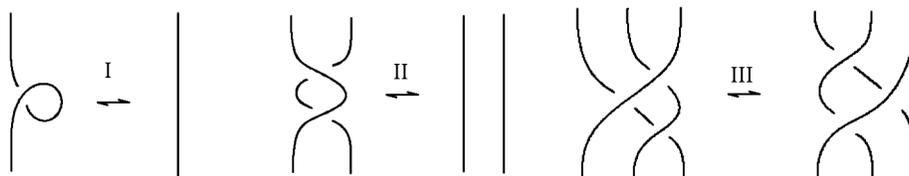


Just as curves can be knotted in \mathbb{R}^3 , surfaces can be knotted in \mathbb{R}^4 ; a *knotted surface diagram* has sheets drawn with breaks to indicate which sheet passes under and which over in the 4th dimension.



Given two knot diagrams, how can we tell whether they represent the same knot or two different knots? A partial answer to this was found in the 1920s by Kurt Reidemeister, who proved the following

Theorem: Two knot diagrams represent ambient isotopic knots if and only if they are related by a sequence of the following moves (the portions of the knot diagram outside the pictured area are identical).



Thus, to prove that two diagrams represent the same knot, we have only to find a sequence of Reidemeister move changing one diagram to the other. For example, the knot diagram below is equivalent to an unknotted circle; we can show this with a sequence of Reidemeister moves.



Exercise: Using Reidemeister moves, change the diagram on the left into the one on the right.



Exercise: Using Reidemeister moves, interchange the components of the *Whitehead link* below.



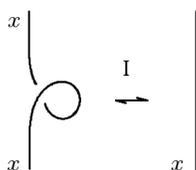
To prove that two diagrams represent the same knot, we just need to find a Reidemeister move sequence changing one diagram to the other. What if we are unable to find such a sequence? Suppose we try without success for 100 hours to change K into K' . What does this tell us? Exactly nothing! It could be that we can't find a sequence because it's really complicated, or it might be that there just is no sequence. How can we prove that two diagrams represent different knots?

To prove that two diagrams are not ambient isotopic, we use *knot invariants*, functions we can evaluate on knot diagrams with the property that the function value is the same before and after all three Reidemeister moves. Then, if our invariant has the same value for two diagrams K and K' , it tells us nothing, but if two diagrams K and K' have *different* values of an invariant, then they can't possibly be related by Reidemeister moves and thus are different knots.

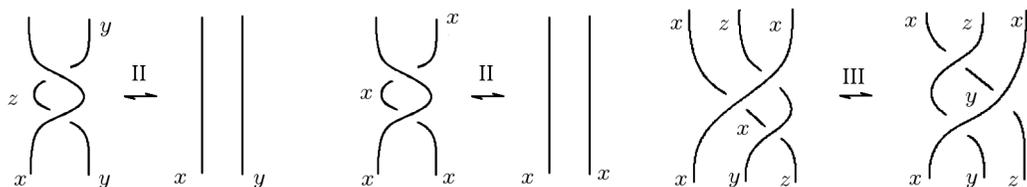
Example: The simplest example of a knot invariant is *Fox coloring*, invented as a homework exercise in an undergraduate course on knot theory by Ralph Fox in the 1950s. A Fox coloring of a knot diagram is a coloring of the arcs in the diagram (the portions from one undercrossing to the next) with one of three colors x, y, z such that

- At every crossing, either all three arcs are the same color or all three are different, and
- At least two colors are used.

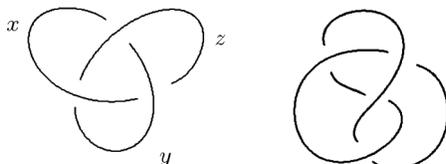
It is an easy exercise to check that if a diagram has a valid Fox coloring before doing a move, there is a unique valid Fox coloring after the move. For instance, in a Reidemeister I move, there is only one color on the arc in the diagram on the right, and two of the arcs meeting in the crossing on the diagram on the left are the same arc, so have the same color; thus the third arc must also have the same color.



For the Reidemeister II move there are two cases; for the III move there are a few more, of which we illustrate one.



We can use this to prove that the trefoil knot 3_1 is not equivalent to the figure eight 4_1 , since the trefoil has the pictured Fox coloring and the figure eight cannot be validly Fox colored (prove this as an exercise!)



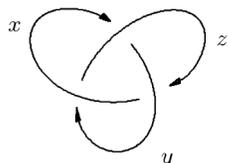
The Alexander Polynomial

We promised connections with linear algebra, so let's see some knot invariants which are defined in terms of linear algebra. Given a knot diagram, choose an *orientation*, i.e. a direction of travel around the knot. Assign variables x_1, x_2, \dots, x_n to each of the arcs. At each crossing, look in the positive direction of the overpass and write down the following equation:

$$\begin{array}{c}
 \begin{array}{ccc}
 & \uparrow & \\
 \hline x_k & & x_i \\
 \hline & | & \\
 & x_j & \\
 \hline
 \end{array} \\
 tx_i + (1-t)x_j = x_k
 \end{array}$$

This recipe gives us a homogeneous system of linear equations. The solution space of this system is known as the *Alexander module* of the knot. Write the system in matrix form $A\vec{x} = \vec{0}$; then the matrix A is the *Alexander matrix* of the knot. In the 1920s, Alexander proved that the determinant of the Alexander matrix is always 0, but perhaps more surprisingly, the $(n-1)$ -minors of the Alexander matrix are always equal up to multiplication by ± 1 and powers of t . We can "normalize" this minor by dividing by powers of t and -1 if needed to get a polynomial with say positive constant term. The resulting polynomial is unchanged by Reidemeister moves and thus is a knot invariant, called the *Alexander polynomial*.

Example: The trefoil has homogeneous system



$$\begin{array}{l}
 tx + (1-t)y = z \\
 ty + (1-t)z = x \\
 tz + (1-t)x = y
 \end{array}
 \Rightarrow
 \begin{bmatrix}
 t & (1-t) & -1 \\
 -1 & t & (1-t) \\
 (1-t) & -1 & t
 \end{bmatrix}
 \begin{bmatrix}
 x \\
 y \\
 z
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 0
 \end{bmatrix}$$

Then we can compute the Alexander polynomial by taking any 2×2 minor. For instance,

$$\begin{vmatrix}
 t & (1-t) \\
 -1 & t
 \end{vmatrix}
 = t^2 - (-1)(1-t) = t^2 - t + 1$$

and

$$\begin{vmatrix}
 t & (1-t) \\
 (1-t) & -1
 \end{vmatrix}
 = -t - (1-t)^2 = -t - (1 - 2t + t^2) = -t^2 + t - 1 \rightarrow t^2 - t + 1.$$

Exercise: Show that the Alexander polynomial of the figure eight knot is $t^2 - 3t + 1$.

Quantum Invariants

For many years the Alexander polynomial was the only knot polynomial (and one of the strongest knot invariants) known. Several times people searched for and found new polynomial knot invariants only to later find that they were merely the Alexander polynomial disguised by a change of variables (e.g., the Conway polynomial). That changed in the mid 1980s when Vaughn Jones (currently at Vanderbilt, until recently at UC Berkeley) was studying the representation theory of objects called *von Neumann algebras* and accidentally discovered a new polynomial knot invariant, now called the *Jones polynomial*. He won the Fields medal (the math version of a Nobel prize) for this.

Soon afterward, groups of mathematicians with the initials HOMFLY (the ‘‘H’’ is Pitzer college’s Jim Hoste) and PT independently showed how to unify the Alexander and Jones polynomials as a single two-variable polynomial knot invariant known as the HOMFLYpt polynomial. In the mid 90s these were shown to arise from a more general construction related to the representation theory of objects known as *quantum groups*, resulting in a family of knot invariants known as *Reshetikhin-Turaev invariants* or *quantum invariants*. To define these, we need matrix multiplication and something called *Kronecker product* or *tensor product*. If A and B are matrices, then the Kronecker product $A \otimes B$ is the block matrix obtained by multiplying each entry of A by a copy of B , i.e.

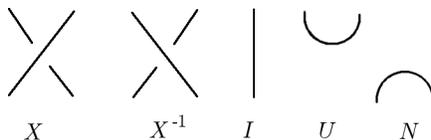
$$\text{if } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pn} \end{bmatrix} \quad \text{then } A \otimes B = \begin{bmatrix} \alpha_{11}B & \alpha_{12}B & \dots & \alpha_{1n}B \\ \alpha_{21}B & \alpha_{22}B & \dots & \alpha_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{p1}B & \alpha_{p2}B & \dots & \alpha_{pn}B \end{bmatrix}.$$

Example:

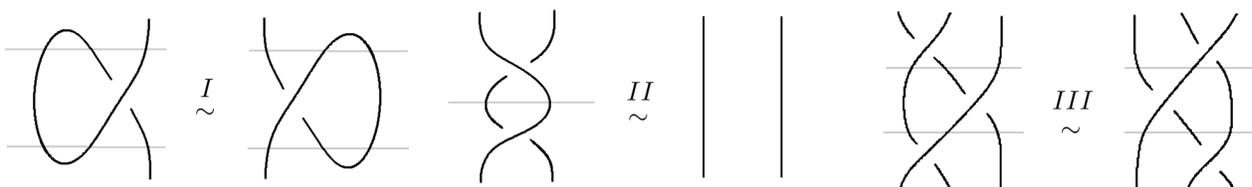
Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$ respectively. Then

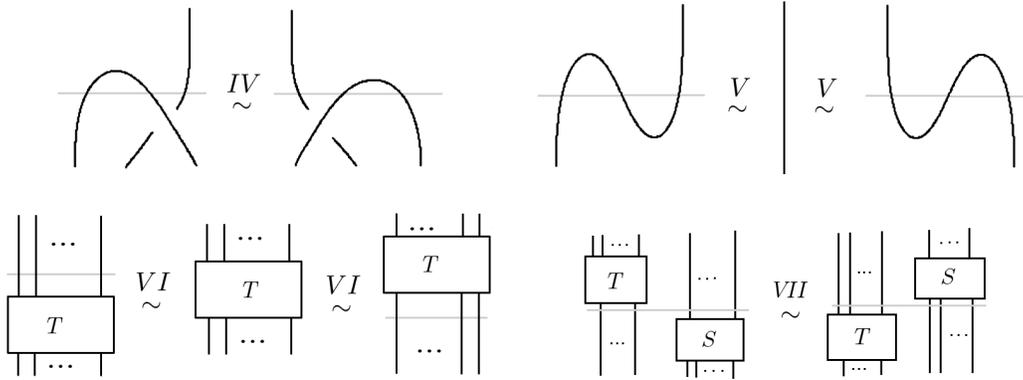
$$A \otimes B = \begin{bmatrix} 1B & 1B \\ 1B & -1B \\ 2B & -1B \end{bmatrix} = \begin{bmatrix} 3 & -1 & 3 & -1 \\ 1 & 2 & 1 & 2 \\ 3 & -1 & -3 & 1 \\ 1 & 2 & -1 & -2 \\ 6 & -2 & 3 & -1 \\ 2 & 4 & -1 & -2 \end{bmatrix}.$$

A *quantum invariant* is a choice of matrices X, X^{-1}, I, U and N representing each of the basic tangles below



such that the conditions imposed by the Reidemeister moves together with a few *planar isotopy moves* are satisfied when we interpret side-by-side stacking as Kronecker product and vertical stacking as matrix multiplication.





For instance move III requires that

$$(X \otimes I)(I \otimes X)(X \otimes I) = (I \otimes X)(X \otimes I)(I \otimes X);$$

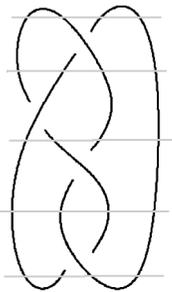
this is known as the *Yang-Baxter equation*.

Any such assignment of matrices satisfying the above conditions defines an invariant of knots by dividing the knot diagram into tangles, replacing the tangles with the matrices, and multiplying out to get a 1×1 matrix, i.e. a scalar (which is a Laurent polynomial since our entries are Laurent polynomials).¹

The Jones polynomial is given by

$$X = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-1} & q - q^{-3} & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 0 \\ -q \\ q^{-1} \\ 0 \end{bmatrix}, \quad N = [0 \quad q \quad -q^{-1} \quad 0].$$

Example: The figure eight knot breaks down as



$$(N \otimes N)(I \otimes X^{-1} \otimes I)(X \otimes I \otimes I)(I \otimes X^{-1} \otimes I)(I \otimes X^{-1} \otimes I)(U \otimes U)$$

resulting in Jones polynomial $q^8 - q^4 + 1 - q^{-4} + q^{-8}$.

¹Technically, we need to make sure our diagram has *writhe* 0, i.e. equal numbers of right-handed and left-handed crossings.