MCMILLAN TYPE THEOREMS ON BOUNDARY BEHAVIOR OF CONFORMAL MAPPINGS.

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Abstract. This paper is an exposition of the McMillan Twist point theorem and "Area" theorem as well as some work of O'Neill and Thurman. In particular we make a simplified presentation of McMillan's original proof of the twist point theorem. The paper is based on talks given at Institut Mittag Leffler and UCLA. The point is to show that the main ideas of the theorems are geometric and potential theoretic.

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1. Introduction

In what follows we discuss two theorems of J.E. McMillan and one due in joint work to R. Thurman and the author on boundary behavior of conformal mappings. Our purpose is to show that all three results are rather intuitive from the point of view of geometry and Brownian motion. This paper is part of a research program to answer the questions that we will pose in the final section. In particular, it is an introduction for [16] where we will give a proof of the twist point theorem which is based only on geometry and properties of Brownian motion. And throughout the paper we will be seeking to simplify arguments to the point where they may be useful in answering higher dimensional versions of the same questions. Hopefully the paper may also serve as an introduction for graduate students to some questions on boundary behavior of conformal mappings and harmonic functions. The paper is mostly self contained with some references to basic classical results. See the end of the introduction for some convenient sources.

McMillan's twist point theorem, proved in [13], says roughly that a typical point on the boundary of a simply connected domain is either an interior tangent point or one about which the domain wraps arbitrarily far in either direction. Conformal mappings enter the picture when we clarify the notion of a typical boundary point. We will use harmonic measure to give a precise statement of the theorem and through out the paper. We give two equivalent definitions. Let Ω be a simply connected domain in $\mathbb{C}$ and $E \subset \partial\Omega$ a Borel set. Let $z_0 \in \Omega$ denote a fixed basepoint.

Definition 1.

$$\omega_{z_0}(E,\Omega) = P\{X_\tau \in E\}$$

where X_t is Brownian motion started at z_0, $\tau = \inf\{t > 0 : X_t \notin \Omega\}$ and P is the Wiener measure on paths.

Thus the harmonic measure $\omega_{z_0}(E)$ (suppressing the reference to Ω) of a Borel subset E of $\partial\Omega$ is the probability that a Brownian traveler started at z_0 exits Ω first through E.

Now let f denote the Riemann mapping of $\mathbb{D}$ onto Ω such that $f(0) = z_0$ and (e.g.) $f'(0) > 0$. Then

Definition 2.

$$\omega_{z_0}(E,\Omega) = \frac{|f^{-1}(E)|}{2\pi}$$

where $|f^{-1}(E)|$ denotes the linear measure of $f^{-1}(E) \subset \partial\mathbb{D}$.

The equivalence of the above two definitions is a theorem of Kakutani from [10]. See [9] for a proof. We can now state the twist point theorem. It is convenient to regard ω_{z_0} as a probability measure on the prime-end compactification of Ω. For example, if Ω is $\mathbb{D} \setminus [0,1]$ then the positive real boundary points each represent two distinct prime ends.

Let Ω and $z_0 \in \Omega$ be as before.

Theorem 1. *With repsect to ω_{z_0}, almost every prime end ζ is either an interior tangent point or has both*

$$\liminf_{z\to\zeta} \arg(z-\zeta) = -\infty$$

and

$$\limsup_{z \to \zeta} \arg(z - \zeta) = +\infty$$

where the limits are taken as $z \to \zeta$ *inside of* Ω *and* $\arg(z-\zeta)$ *denotes a continuous branch of the argument defined in* Ω*.*

We will give a detailed proof of this result along the lines in McMillan's original paper but with as much simplification as possible. In particular we will see that the original argument is quite natural from the point of view of Definition 2 and that in this sense the proof boils down to a combination of Plessner's theorem, intuition with Brownian motion and figure 1.

Around the same time, in [12], McMillan also proved another nice result (referred to above as the "Area Theorem") on the almost sure character of boundary points of simply connected domains. To describe it we will need some notation. See figure 1. Fix $z_0 \in \Omega$ and for each ideal accessible boundary point $\zeta \in \partial\Omega$ and $\rho < |z_0 - \zeta|$ let $\gamma(\zeta, \rho) \subset \{z : |z - \zeta| = \rho\}$ be the circular crosscut of Ω separating ζ from z_0 that can be joined to ζ by a Jordan arc contained in $\Omega \cap \{z : |z - \zeta| < \rho\}$. Let $L(\zeta, \rho)$ denote the Euclidean length of $\gamma(\zeta, \rho)$ and let

$$A(\zeta, r) = \int_0^r L(\zeta, \rho)\, d\rho.$$

McMillan proved the following

Theorem 2. *With respect to the harmonic measure, almost every ideal accessible boundary point* ζ *satisfies*

$$\limsup_{r \to 0} \frac{A(\zeta, r)}{\pi r^2} \geq \frac{1}{2}.$$

In words, the theorem states loosely that at almost every boundary point there is some sequence of scales at which the domain looks locally as fat as

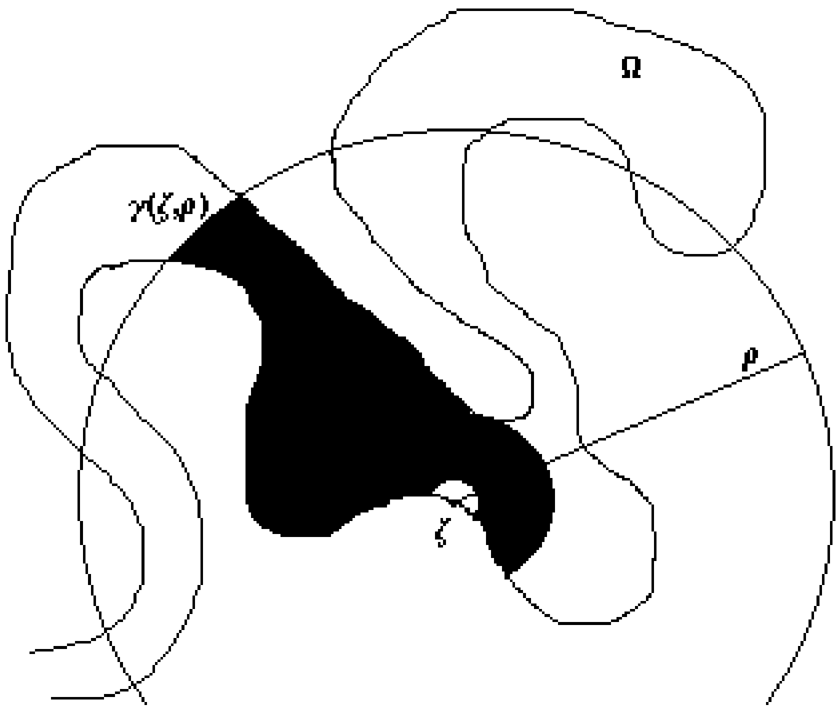

FIGURE 1. An area swept out by the arcs in the definition of $L(\zeta, \rho)$

a semi-disk. At the end of [12] the question of whether

$$\liminf_{r\to 0} \frac{A(\zeta, r)}{\pi r^2} \leq \frac{1}{2} \qquad \text{a.e.-}\omega$$

is posed. The positive answer (Theorem 3) was found in [17] and [18]. Here we will only outline the ideas involved with the aim of showing that the proofs of theorems 2 and 3 boil down to Brownian motion and respectively, figures 7 and 8. Also included in [12] is an example of a domain for which

$$\limsup_{r\to 0} \frac{A(\zeta, r)}{\pi r^2} = 1 \qquad \text{a.e.-}\omega$$

and

$$\liminf_{r\to 0} \frac{A(\zeta, r)}{\pi r^2} = 0 \qquad \text{a.e.-}\omega$$

In section 2 we will prove two lemmas of McMillan which contain the intuition about Brownian motion that we will need. In section 3 we will give a short discussion and definitions of cone points, tangent points and twist points. Sections 4, 5 and 6 contain the discussions of theorems 1, 2 and 3 respectively. In the final section we record some related open questions in two and higher dimensions.

For background on prime ends, Plessner's theorem and a nice analytical proof of the twist point theorem see [19] and [20]. For Fatou's theorem see [8] and for the Beurling projection theorem and much more about harmonic measure see [1] and [9]. For the connection between Brownian motion and harmonic measure see [9] and [2] and the paper [14].

2. Zero harmonic measure and escape routes for Brownian motion

The lemmas in this section are the main technical tool employed in [13] and [12]. The purpose of the discussion here is to show that they are quite natural from the Brownian motion point of view. Loosely speaking they state that in order to prove that a subset E of the boundary of a simply connected domain has zero harmonic measure, it is enough to show that almost any Brownian traveler on its way to E within the domain has infinitely many opportunities to escape to $\partial\Omega \setminus E$ with some positive probability.

The simplest version of this principal was used in [12]. Let Ω denote a simply connected domain in the plane and let $z_0 \in \Omega$ be a fixed basepoint. If S_1 and S_2 are subsets of $\overline{\Omega}$ let $\mathrm{dist}_\Omega(S_1, S_2) = \inf(\mathrm{diam}\,\gamma)$ where the infimum is over Jordan arcs γ which lie in Ω and join S_1 and S_2.

Lemma 2.1. *Let $E \subset \partial\Omega$ consist of accessible boundary points and suppose that for each $a \in E$ there exists a sequence $\{c_n\}$ of crosscuts of Ω each of which separates a from z_0 and for which both* $\mathrm{diam}\, c_n \to 0$ *and*

$$\sup_n \frac{\mathrm{diam}\, c_n}{\mathrm{dist}_\Omega(c_n, E)} < \infty.$$

Then $\omega_{z_0}(E, \Omega) = 0$.

Proof. We may assume that

$$\sup_n \frac{\operatorname{diam} c_n}{\operatorname{dist}_\Omega(c_n, E)} < K < +\infty.$$

If $z \in c_n$, Harnack's inequality shows that $\omega_z(E^c \cap \partial\Omega, \Omega) > c(K) > 0$. Then by conformal invariance and lemma 2.3 we are done. We will "sketch" a proof which does not rely on conformal invariance.

Assume as before that

$$\sup_n \frac{\operatorname{diam} c_n}{\operatorname{dist}_\Omega(c_n, E)} < K < +\infty.$$

The set of boundary points of Ω which are not accessible by a rectifiable arc within Ω has harmonic measure zero. This is a purely potential theoretic fact and is true in higher dimensions as well. See [5]. (It is also easy to see by considering the area of Ω as an integral of $|f'|^2$ where f is the conformal mapping.) So a compact $K \subset \partial\Omega$ is the countable union of sets which are compact in the $\operatorname{dist}_\Omega$ topology on $\overline{\Omega}$ and a set of zero harmonic measure.

Assuming that the set E is compact in the $\operatorname{dist}_\Omega$ topology we can cover E by "generations" of the curves c_n as in figure 2. Note that the width of each escape route is comparable to its diameter. Harnack's principle and the strong Markov property then finish the proof. (See [15] for the details.) □

The stronger version of the lemma which was used in [13] requires Stolz cones. Fix $0 < r < 1$. Given $e^{i\theta} \in \mathbb{D}$, let $\Gamma_\alpha(e^{i\theta})$ be the convex hull of $\{z : |z| = r\}$ and $\{e^{i\theta}\}$ where

$$\alpha(r) = 2 \arctan \frac{r}{\sqrt{1-r^2}}$$

is the angle at the vertex $e^{i\theta}$. Suppressing the reference to r we have defined our Stolz cones $\Gamma_\alpha(e^{i\theta})$. Let $f : \mathbb{D} \to \Omega$ be a conformal mapping and let $E \subset \partial\Omega$ consist of accessible boundary points.

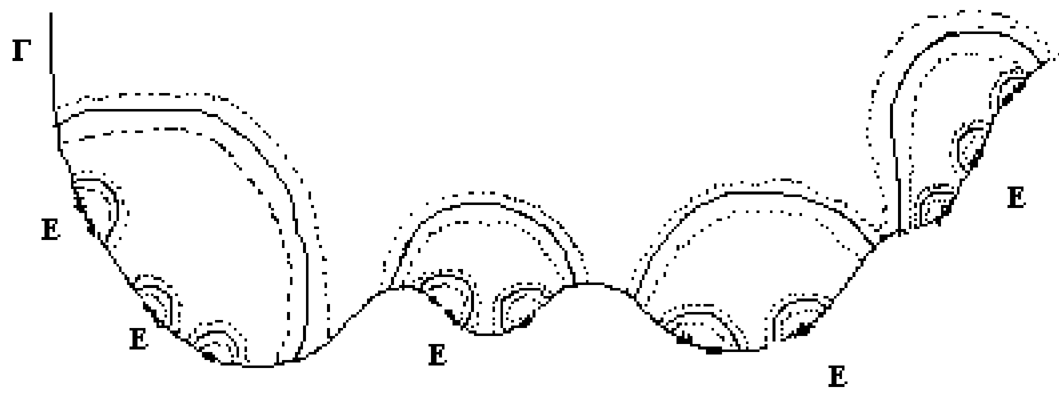

FIGURE 2. Two generations of escape routes for Brownian motion from lemma 2.1

Lemma 2.2. *Suppose that for each $e^{i\theta} \in f^{-1}(E)$ there is a sequence of points $z_n \to e^{i\theta}$ in $\mathbb{D}$ such that the z_n are contained in some open triangle in $\mathbb{D}$ and that for each z_n there is a Jordan arc $A_n \subset \Omega$ with one endpoint $f(z_n)$ and the other on $\partial\Omega$ such that*

$$\sup_n \frac{\text{diam } A_n}{\text{dist}_\Omega(A_n, E)} < \infty.$$

Then $\omega(E, \Omega) = 0$.

Proof. Dividing E into countably many pieces we may assume that the points z_n lie in the union of all the cones $\Gamma_\alpha(e^{i\theta})$, $e^{i\theta} \in f^{-1}(E)$ for some fixed α. In the same way we may assume that

$$\sup_n \frac{\text{diam } A_n}{\text{dist}_\Omega(A_n, E)} < K < \infty.$$

By Harnack's principle there is $c(K) > 0$ such that

$$\omega_{z_n}(E^c \cap \partial\Omega, \Omega) > c(K) > 0$$

at each z_n. By conformal invariance and lemma 2.3 we are done.

□

Let $E \subset \partial\mathbb{D}$, $0 < \alpha < \pi$ and $\epsilon > 0$.

Lemma 2.3. *Suppose that for each $e^{i\theta} \in E$ there exists a sequence $z_n \to e^{i\theta}$ with $\{z_n\} \subset \Gamma_\alpha(e^{i\theta})$ such that*

$$\omega_{z_n}(E, \mathbb{D}) < 1 - \epsilon.$$

Then $\omega(E) = 0$.

Proof. By Fatou's theorem $u(z) = \omega_z(E, \mathbb{D})$ has the nontangential limit 1 at almost every point of E. Thus the conditions of the theorem imply that $|E| = 0$. □

The Brownian motion proof of lemma 2.3 would run along the following lines. For each α and for almost every Brownian path η which hits $\partial\mathbb{D}$ first in E at time $\tau(\eta) = \inf\{X_t(\eta) \notin \mathbb{D}\}$ there is $t_0(\eta) < \tau(\eta)$ such that

$$X_t(\eta) \in \bigcup_{e^{i\theta} \in E} \Gamma_\alpha(e^{i\theta}), \qquad t_0 \le t < \tau.$$

This follows from a stopping time argument on crossings between $\cup\Gamma_\alpha$ and $(\cup\Gamma_{2\alpha})^c$ and the strong Markov property. Thus almost every such Brownian path passes near infinitely many of the z_n, experiencing at each pass a probability greater than $c(K) > 0$ of exiting $\mathbb{D}$ through $E^c \cap \partial\mathbb{D}$. By the Strong Markov property again, $P\{X_\tau \in E\} = 0$.

To prove lemma 2.2 without relying on conformal invariance we can, for example, use the ideas in the paper of Brelot and Choquet, [5], to define the analog of cones in Ω via Green lines. See figure 3. Fix $z_0 \in \Omega$ and let $\zeta \in \partial\Omega$. Given $z \in \Omega$ let ℓ_0 denote the Green's line from z_0 to z (neglecting countably many critical points). Consider the Green lines starting at z whose tangents at z sweep out an angle α centered at the tangent of ℓ_0 at z. If ζ is one of the endpoints on $\partial\Omega$ of such a Green line then put $z \in \Gamma_\alpha(\zeta)$. By the work of Brelot and Choquet, the set of such endpoints of Green lines has $\omega_z = \frac{\alpha}{2\pi}$. The same Brownian motion argument outlined above for the disk can now be made directly in Ω with the $\Gamma_\alpha(\zeta)$ playing the role of the cones.

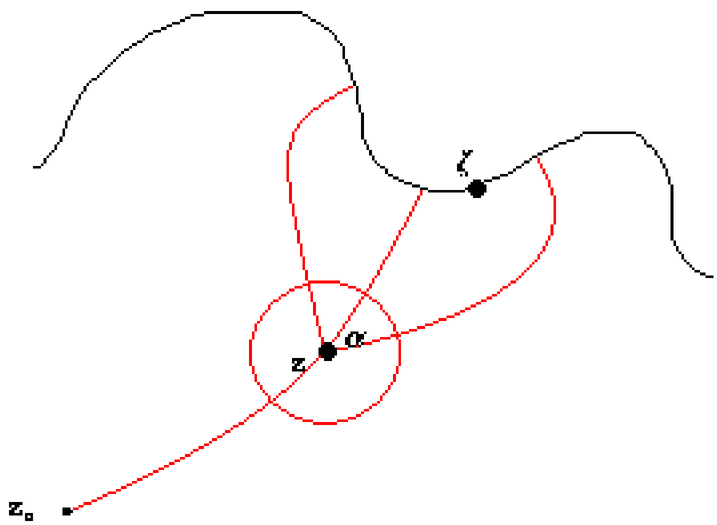

FIGURE 3. The point z is contained in $\Gamma_\alpha(\zeta)$, the cone of aperture α at ζ

3. Cone points, tangent points, conformality.

In preparation for our discussion of the twist point theorem in Section 4 we make the following definitions.

Definition 3. *A point $\zeta \in \partial\Omega$ is an inner tangent to $\partial\Omega$ if there exists exactly one $0 \leq \phi_0 < 2\pi$ such that for each $0 < \epsilon < \frac{\pi}{2}$ there is a $\delta > 0$ so that*

$$\{\zeta + \rho e^{i\phi} : 0 < \rho < \delta, |\phi - \phi_0| < \frac{\pi}{2} - \epsilon\} \subset \Omega.$$

Definition 4. *The boundary point ζ is called a cone point if it is the vertex of some open triangle contained in Ω.*

Definition 5. *The conformal mapping f from $\mathbb{D}$ onto Ω is said to be conformal at the boundary point $e^{i\theta}$ if the non-tangential limit of $f'(z)$ at $e^{i\theta}$ exists and is not 0 or ∞.*

With respect to harmonic measure on $\partial\Omega$, almost every cone point is an inner tangent point. This is yet another result from [13]. We note that a proof depends only on the construction of the correct Lipschitz sub-domains of Ω and Harnack's inequality. (See [9].) Moving to the disk, let

$$\mathrm{Sect}(f) = f^{-1}(\{\text{Cone Points}\}).$$

Then f is conformal at almost every point of Sect(f). (See [20] chapter 4.)

It is also convenient to pull the notion of twisting back to the disk.

Definition 6. *The conformal mapping f from $\mathbb{D}$ onto Ω is twisting at $e^{i\theta}$ if as z approaches $e^{i\theta}$ within a Stolz cone at $e^{i\theta}$,*

$$\liminf_{z \to e^{i\theta}} \arg(f(z) - f(e^{i\theta})) = -\infty$$

and

$$\limsup_{z \to e^{i\theta}} \arg(f(z) - f(e^{i\theta})) = +\infty.$$

The twist point theorem can now be stated as follows. At almost every $e^{i\theta} \in \mathbb{D}$ either f is conformal or f is twisting.

We now recall Plessner's theorem. (See [20] chapter 6.) It is a consequence of the theorem of Fatou used in lemma 2.3 together with another construction of Lipschitz subdomains and the conformal invariance of harmonic measure. Let g be a meromorphic function in $\mathbb{D}$.

Theorem 3.1. *For almost every $e^{i\theta}$ either g has a finite angular limit at $e^{i\theta}$ or for any $0 < \alpha < \pi$, $g(\Gamma_\alpha(e^{i\theta}))$ is a dense subset of the plane.*

An immediate consequence of Plessner's theorem is the Privalov uniqueness theorem.

Theorem 3.2. *If the meromorphic function g is not identically zero then it has the angular limit zero almost nowhere on $\partial\mathbb{D}$.*

Applying Plessner's theorem to the function $\log f'$ we see that the failure of conformality at $e^{i\theta}$ is almost everywhere equivalent to the unboundedness above and below of $\arg f'$ in each Stolz cone $\Gamma_\alpha(e^{i\theta})$. Thus to prove the twist point theorem it is enough to show that unboundedness from above and below of $\arg f'$ is almost everywhere equivalent to the condition that f is twisting.

In the next section we will show that the set E where $\arg f'$ is unbounded from above and below in Stolz cones and at the same time

$$\limsup_{z \to e^{i\theta}} \arg(f(z) - f(e^{i\theta})) < +\infty,$$

has harmonic measure zero. Since the same reasoning applies with lim sup replaced by lim inf, that will prove the theorem.

4. The twist point theorem

We will now give a proof of the twist point theorem following McMillan's original argument. Our purpose is to show that the argument is physically intuitive and to simplify matters wherever possible. The general strategy is depicted in figure 4. A curve along which $\arg f'$ grows without bound is forced to wind many times around many boundary points, thus excluding them from membership in the set E and making them good escape destinations as in lemma 2.2. In [16] we will give another proof of the twist point theorem which does not use any conformal mapping. Then the main task is to make the same argument directly using the failure of the cone point condition, i.e. to show that as Brownian motion approaches a set of positive harmonic measure which fails the cone condition it is forced to see large (in the sense of harmonic measure) parts of the boundary about which the domain winds by a large amount.

The proof below will be done in four steps. In the first step we will reduce the set E by chopping it into countably many pieces. In step 2 we will see that the reduction in step 1 allows us to show that the curve σ shown in figure 4 starting at $w(0)$ is forced to wind around w^*, a later point on the curve σ.

In the third step we will see that the point w^* can be chosen so that a path from w^* to the boundary of Ω can be found which lies in a half plane

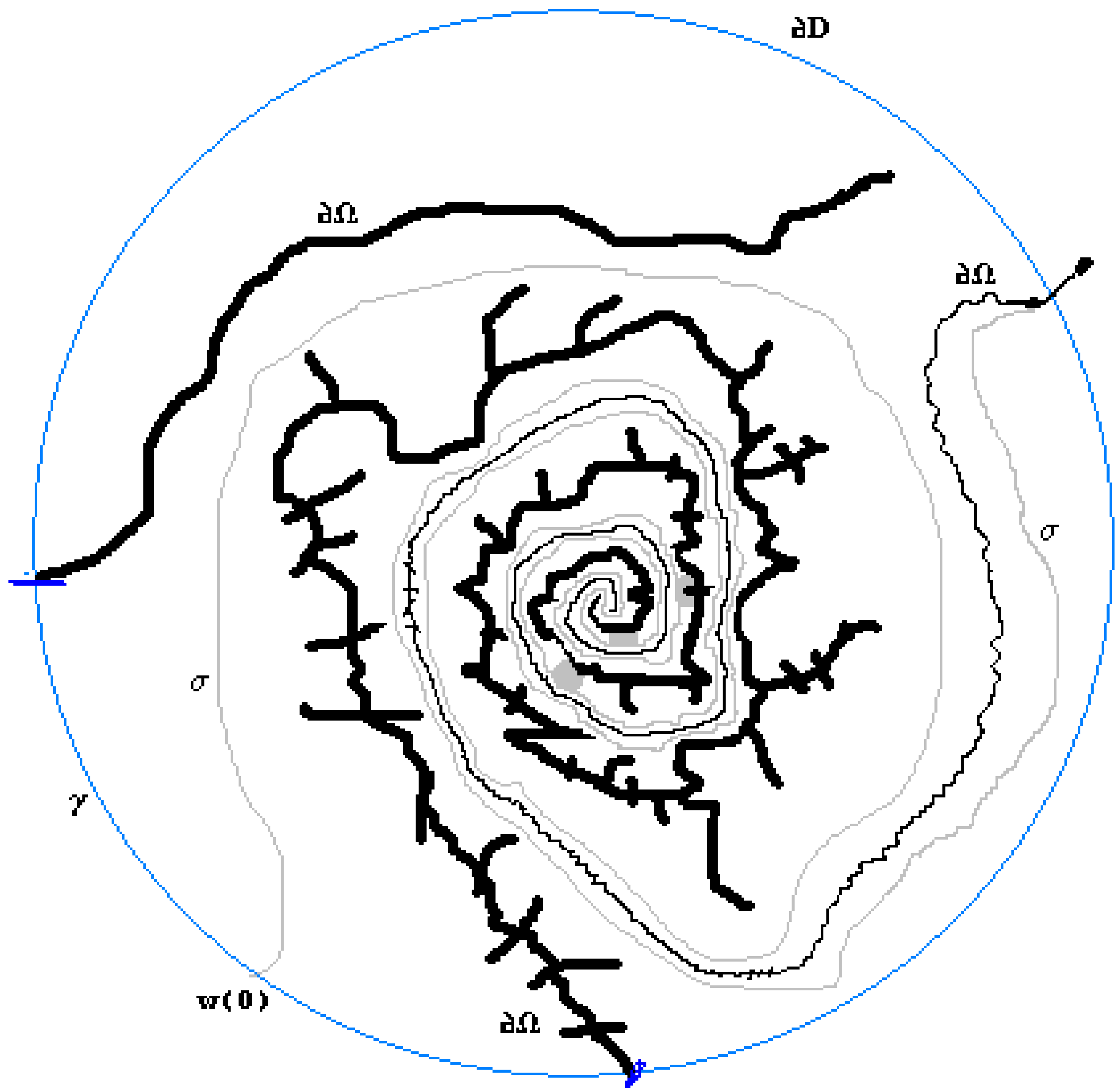

FIGURE 4. A large increase in arg f' implies that a path winds around many boundary points. Some possible escape routes are shown in gray. The point w^* is on the curve σ near the center of the spiral.

who's boundary contains w^*. This already makes it evident, as in figure 4, that there must be boundary within the wound up parts of σ but in step 4 we will make this precise by showing how step 3 lets us apply lemmas 2.2 and 2.3 to finish the proof.

We come to the details.

Step 1 Let $E \subset \mathbb{D}$ denote the set of $e^{i\theta}$ where $\arg f'$ is unbounded above on $\Gamma_\alpha(e^{i\theta})$ but $\limsup_{z\in\mathbb{D}} \arg(f(z) - f(e^{i\theta})) < +\infty$. We want to show that $\omega(E) = 0$. Let D denote a disk with rational center and radius and let Ω_D be one of the countably many components of $D \cap \Omega$. Dividing E into countably many pieces, we may assume

(i) $f(z) \in \Omega_D$ if $z \in \Gamma_\alpha(e^{i\theta})$ is sufficiently close to $e^{i\theta} \in E$.

(ii) $f(D_{r(\alpha)}(0)) \cap D = \emptyset$

(iii) $\mathrm{Arg}(w - f(e^{i\theta})) - \mathrm{Arg}(w_0 - f(e^{i\theta})) \leq M < +\infty$ for each $w \in \Omega_D$, each $w_0 \in \partial\Omega_D \cap \Omega$ and each $e^{i\theta} \in E$.

For (iii), note that Ω_D is a simply connected subset of the disk D so that all values of $\mathrm{Arg}(w_0 - f(e^{i\theta}))$ for $w_0 \in \partial\Omega_D \cap \Omega$ lie in some interval of length 2π.

Step 2

Choose z_0 such that $f(z_0) \in \partial\Omega_D \cap \Omega$ and fix $e^{i\theta} \in E$. Let $z^* \in \Gamma_\alpha(e^{i\theta})$ be a point such that

$$\arg f'(z^*) - \arg f'(z_0) > 2M.$$

Let $z(t) = (1-t)z_0 + tz^*$, $w(t) = f(z(t))$ and $\sigma = \{w(t) : 0 \leq t \leq 1\}$. Moving z_0 and z^* if necessary, we may assume that $\sigma \subset \Omega_D$.

The function $g(\tau, t) = w(t) - w(\tau)$ is continuous and nowhere zero on $T = \{(\tau, t) : 0 < t \leq 1, 0 \leq \tau < t\}$. So we may determine a branch of the argument so that $\phi(\tau, t) = \arg(w(t) - w(\tau))$ is continuous on T. Because $w(0) \in \partial\Omega_D \cap \Omega$ and $\sigma \subset \Omega_D$, we can determine the branch so that

$$0 \leq \phi(0, t) \leq 2\pi$$

for sufficiently small t and therefore so that

$$-\pi \leq \phi(0, t) \leq 3\pi$$

for all $0 \leq t \leq 1$. Since $w'(t)$ is continuous and never zero

$$\phi(t_0) = \lim_{(\tau,t)\to(t_0,t_0)} \phi(\tau, t)$$

exists for each $0 \leq t_0 \leq 1$. Thus $\phi(t)$ is continuous and $\phi(t) - \arg f'(z(t))$ is constant. It follows that

$$\phi(1) - \phi(0) = \arg f'(z^*) - \arg f'(z_0).$$

Since both

$$-\pi \leq \phi(0,1) \leq 3\pi$$

and

$$-\pi \leq \phi(0) \leq 3\pi$$

it follows that

$$\phi(1) - \phi(0,1) \geq \arg f'(z^*) - \arg f'(z_0) - 4\pi.$$

But $\phi(1) - \phi(0,1)$ is the change in $\phi(\tau, 1)$ as τ increases from 0 to 1. So the curve σ winds around $w^* = f(z^*)$ by an angle of at least $2M - 4\pi$.

Step 3

Let S denote the component of the level curve of $\arg f'$ which contains z^*. We may assume that S contains no critical point of $\arg f'$. By applying the maximum principle to $\log |f'|$ we see that S has exactly two endpoints on $\partial\mathbb{D}$ which we label a and b. By the Privalov uniqueness theorem and by increasing M if necessary, we may assume that S is contained in a disk of radius $1/2$ centered at $e^{i\theta}$. Then we may also assume that in moving along the level curve from a to b that z^* is the last point on S contained in $\overline{\Gamma_\alpha(e^{i\theta})}$. Let A denote the piece of S running from z^* to b. (See figure 5.) We claim that $f(A)$ is contained in a half plane whose boundary contains $w^* = f(z^*)$. To see this let $z(t)$, $0 \leq t \leq 1$ be a parameterization of A with $z(0) = z^*$ and $z(1) = b$. The function $\arg f'(z(t))$ is constant and we may assume that $\log |f'(z(t))|$ is decreasing. Let H denote the half plane containing b whose boundary contains z^* and $e^{i\theta}$.

Move H to the upper half plane by $\zeta = \eta z + \nu$ with $\zeta(z^*) = 0$ and $|\eta| = 1$. Let $F(\zeta) = f(\frac{\zeta - b}{a})$ and $\zeta(t) = u(t) + iv(t) = \eta z(t) + \nu$. Now $\arg F'(\zeta(t)) = \lambda$ is constant and $|F'(\zeta(t))|$ is decreasing in $0 \leq t \leq 1$.

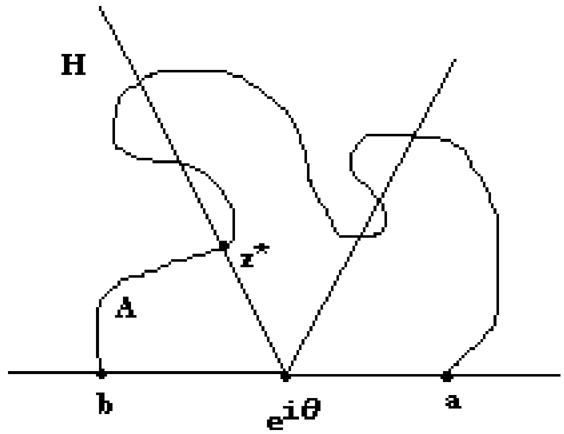

FIGURE 5. Step 3

We have

$$F(\zeta(t)) - F(0) = e^{i\lambda}\left(\int_0^t |F'(\zeta(t))|\,du(t) + i\int_0^t |F'(\zeta(t))|\,dv(t)\right).$$

Because $v(t)$ is non-negative and $|F'(\zeta(t))|$ is decreasing we have

$$\int_0^t |F'(\zeta(t))|\,dv(t) = |F'(\zeta(t))|v(t) - \int_0^t v(t)\,d|F'(\zeta(t))| \geq 0$$

and the claim is proved.

Step 4 It remains to check that the curve σ has a winding angle bigger than M around a set of boundary points which has large ω_{w^*} measure. This is practically evident from step 3. We will give a proof mostly via figure 6. The following list of notations should be checked off on the figure.

Let $z(t)$ be the parameterization of A from step 3 and $w(t) = f(z(t))$. Let $d = \mathrm{dist}(w^*, \partial\Omega)$,

$$D_{w^*} = \{w : |w - w^*| < d\},$$

$$t^* = \sup\{0 \leq t \leq 1 : w(t) \in D_{w^*}\},$$

$$p = w(t^*).$$

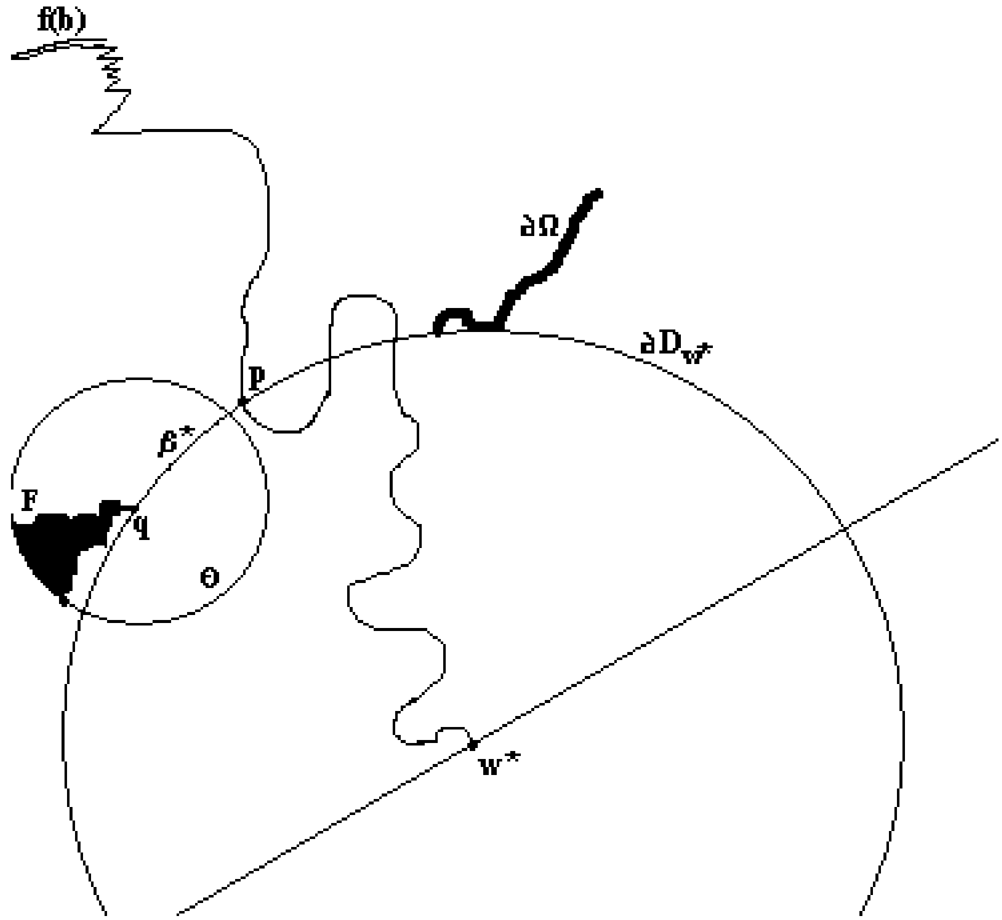

FIGURE 6. Step 4

Let β be the component of $\partial D_{w^*} \cap \Omega$ which contains p and let q be the nearer endpoint of β to p. Let β^* be the part of β connecting p and q. Let $D_q(d/4)$ denote the disk of radius $d/4$ centered at q. Then $\partial D_q(d/4) \cap \Omega$ has exactly one connected component arc which intersects D_{w^*}. Denote it by Θ. Let F be the set of prime ends separated from w^* by Θ and let q^* be some point of F. We claim that the curve σ winds around q^* nearly as much as it does around w^*.

The winding of σ around w^* is within 4π of its winding around any other point of $f(A) \cup \beta^* \cup F$ by step 3. For the winding angle of σ about w^* to exceed the winding around q^* by more than 8π, σ would have to cross β^* twice. By deleting appropriate parts of σ and replacing them with arcs of β^* we could then construct a closed curve in Ω whose interior contains $f(A)$. This is a contradiction because Ω is simply connected. The argument is simpler if $f(b) \in \partial D_{w^*}$.

By the Beurling projection theorem,(e.g. See [1] or [9]), $\omega_{w^*}(F) > c > 0$ where c is some universal constant. So if M is sufficiently large, then $F \subset E^c$. An appeal to lemma 2.3 now completes McMillan's proof of the twist point theorem.

5. McMillan's Area theorem

The paper [15] contains a shortened but detailed presentation of the proof of McMillan's area theorem from [12]. Here we will only give the main ideas of the proof together with figure X which illustrates the construction of the required escape routes for Brownian motion. We note that the proof needs no mention of conformal mappings.

Let $w_0 \in \Omega$ be a fixed base point and let E denote the set of accessible boundary points a for which

$$\limsup_{r\to 0} \frac{A(a,r)}{\pi r^2} < \frac{1}{2}.$$

Theorem 2

$$\omega(E) = 0$$

Proof.

Let $w_0 \in \Omega$ be a fixed base point and let $C_1, C_2, \ldots$ denote positive constants which are either numerical or depend only on m and k. If we use the phrase "bounded below" or "bounded away from" it will also mean by a constant depending only on m and k. As in the proof of the twist point theorem we gain control by breaking E into countably many pieces $E = \cup E_{m,k}$ where

$$E_{m,k} = \{a : \frac{A(a,r)}{\pi r^2} < \frac{1}{2} - \frac{1}{m}, \quad \forall r < \frac{1}{k}\}$$

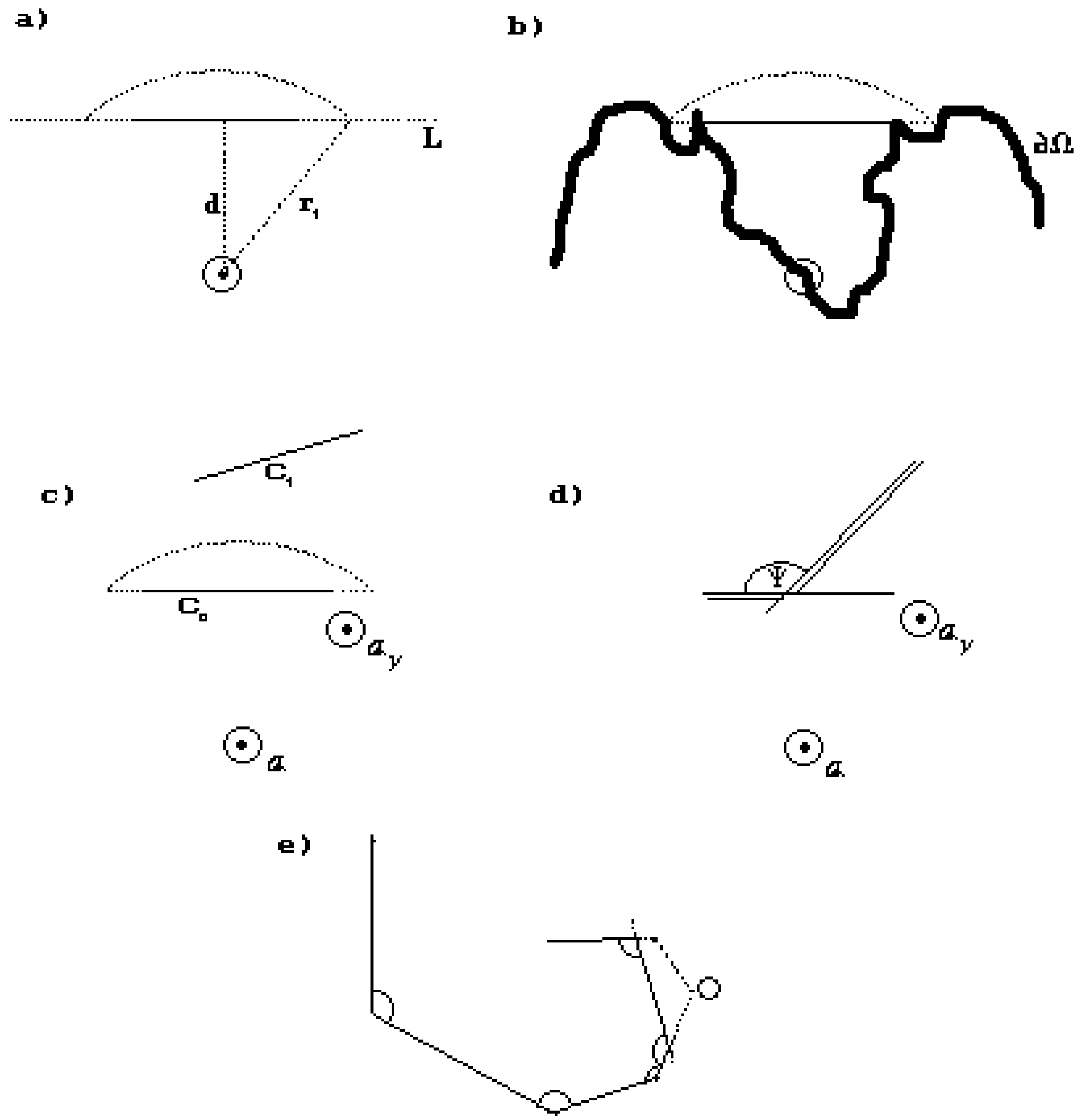

FIGURE 7. The proof of theorem 2.

As mentioned in the proof of lemma 2.1, it is enough to then show that if $F \subset E_{m,k}$ is compact in the dist_Ω topology then $\omega(F) = 0$. Given such a set F we will show that for any $a \in F$ and $\epsilon > 0$ there is a crosscut c of Ω with diameter less than $C_1\epsilon$ separating a from the base point w_0 such that

$$\operatorname{diam} c < C_2 \operatorname{dist}_\Omega(c, F).$$

By lemma 2.1 or 2.2 then, we will be done.

There is a $1 > C_3 > 0$ such that for any $r < \frac{1}{k}$ and $a \in E_{m,k}$ the set $\{r' \in (C_3 r, r) : L(a, r') < (\frac{1}{2} - \frac{2}{m})2\pi r'\}$ has positive Lebesgue measure. To any point $a \in F$ and any $\frac{1}{k} > \epsilon > 0$ we associate a single line segment which is a crosscut of Ω as pictured in figure 7a) and 7b). In the figure, $C_3\epsilon < r' < \epsilon$, $L(a, r') < (\frac{1}{2} - \frac{2}{m})2\pi r'$ and $d > C_4\epsilon$. The segment is chosen so that it can be joined to a by a Jordan curve in Ω which does not touch any other part of the line L or the circle $\{z : |a - z| = r'\}$. We will refer to such a segment below as a basic segment at a.

We will now construct the required escape route for a given point $a \in F$ at a given scale $\epsilon > 0$.

Let $h = \frac{\delta}{10} \sin(\frac{2}{m})$. Cover F with finitely many "disks " Δ_ν (in the metric $\operatorname{dist}_\Omega$) of relative diameter less than $h\epsilon$. Assume that $a \in \Delta_1$ and construct the basic segment c_0 for a. If $\operatorname{dist}_\Omega(c_0, F) \geq h\epsilon$ we are done. Otherwise the construction continues. Choose a disk Δ_ν such that $\operatorname{dist}_\Omega(c_0, \Delta_\nu \cap F) < h\epsilon$ and a point $a_\nu \in F \cap \Delta_\nu$ and construct a basic segment for a_ν. Two possibilities for this second step are shown in figure 7c) and 7d). In c) the second stage arc is c_1. In d) it is the doubled arc. Note that in d) the angle of intersection Ψ is bounded away from π. The construction continues until no disks are within $h\epsilon$ of the constructed polygonal arc. At the n^{th} stage a polygonal arc c_n has been constructed. If there is a disk Δ_ν within $h\epsilon$ of c_n in the relative distance, select one and construct the basic segment at some point of $F \cap \Delta_\nu$. There are three possibilities to consider depending on whether this new basic segment intersects c_n in 0, 1 or more than 1 points.

In the 0 intersection case, the situation is as in figure 7c). The new arc c_{n+1} is simply the basic segment for the offending disk at the n^{th} stage. It is easy to show that a is separated from w_0 by this arc.

In case of more than one intersection, it takes a brief topological argument to show that there can be at most two intersections. Then the situation is as in figure 7e). Note that all the angles pictured are bounded by $\Psi < \pi$.

In the one intersection case denote the new basic segment by c^*. Then the new arc c_{n+1} is the union of the piece of c_n on the w_0 side of c^* and the piece of c^* on the w_0 side of c_n. Again it takes a short argument to verify that c_{n+1} separates a from w_0. The control on the angles of intersection at all steps by the angle $\Psi < \pi$ ensures that the diameter of the constructed cross cut is bounded by $C_4\epsilon$ at every stage. After exhausting the finite supply of disks or stopping earlier because the disks are far enough away, we are done. □

We refer to [15] or [12] for the details. (There is a typo in [15] for the one intersection case. In line 6 from the bottom of page 233 a^* should be c_0^*.)

6. Another theorem of the same type.

In [17] and [18] the positive answers to the questions asked in [12] are given in detail. Our aim here is again to show that the heart of the matter is a simple geometric idea together with intuition about Brownian motion or Harnack's inequality.

Let $w_0 \in \Omega$ be a fixed base point and let E denote the set of accessible boundary points such that

$$\liminf_{r\to 0} \frac{A(a,r)}{\pi r^2} > \frac{1}{2}.$$

Theorem 3

$$\omega(E) = 0$$

Proof. We again break the set in question into countably many pieces.

$$E_{m,k} = \{a : \frac{A(a,r)}{\pi r^2} > \frac{1}{2} + \frac{1}{m} \quad \forall r < \frac{1}{k}\}$$

but now the strategy of the proof is different. As before, let $C_1, C_2, \ldots$ denote constants depending only on m or k or which are purely numerical.

We will show that the geometric condition describing points of $E_{m,k}$ is incompatible with the existence of a point of density of $f^{-1}(E_{m,k})$, where f is a conformal mapping from $\mathbb{D}$ to Ω. In fact we show that there is $\delta(m,k) > 0$ such that if $\omega(z_0, E_{m,k}) > 1 - \delta$ for some $z_0 \in \Omega$, then it will be possible to construct a closed curve in Ω surrounding $z_1 \in \partial\Omega$, a nearest boundary point to z_0. This will contradict the simple connectivity of Ω.

Referring now to figure 8, suppose the sub-region of Ω consisting of annular corridors has been constructed as shown. It is assumed to extend, at the current stage, by an angle of θ_n around z_1 and to have a thickness bounded from below by $C_1|z_0 - z_1|$. The lower bound on the thickness gives Brownian travelers starting at z_0 a probability $>> \delta$ of reaching boundary near z_n. Assume also that in each previous stage of construction our corridor has been further extended by at least the angle $\alpha(m) > 0$ around z_1. The density condition $\omega(z_0, E_{m,k}) > 1 - \delta$ implies that there are many points of $E_{m,k}$ near z_n. The precise argument uses only Harnack's inequality and the Beurling projection theorem. We claim that the large harmonic measure of points of $E_{m,k}$ near z_n together with the geometric condition on the points allows us to proceed to the next step of the construction.

Notice that in figure 8 the piece of corridor labelled S is different than the ones above it. The pieces above S represent the inductive step which we are about to describe. It is the inductive step which assures our path turns by some fixed additional amount, $\alpha(m) > 0$, around z_1. The inclusion of S is a technical neccessity which prevents the constructed corridor from turning the wrong way after several steps. In [18] we simply included an S step in each stage of the construction and we indicate how to do this below.

We are assuming that we have a subset $J \subset E_{m,k}$ with ω_{z_0} measure larger than $C_2 > 0$ near z_n at the end of the currently constructed corridor.

Let r_1 be as shown and recall the definition of $\gamma(\zeta, r)$ from the introduction. We now further assume that if $\zeta \in J$ then for any $r < r_1$, the arc

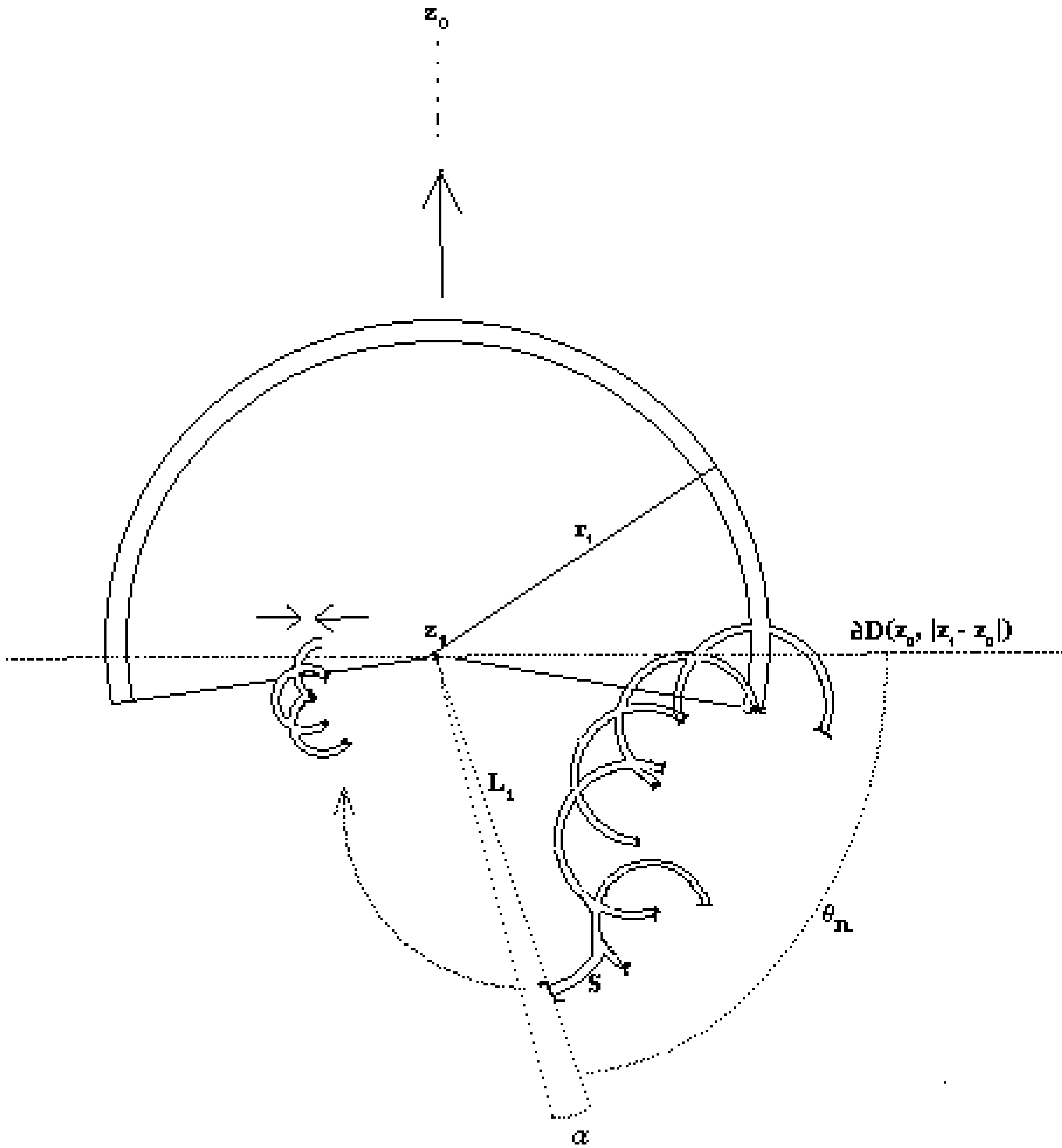

FIGURE 8. The proof of theorem 3.

$\gamma(\zeta, r)$ intersects the whole previously constructed pathway. If this is not true for some $\zeta \in J$ and some $r < r_1$ then we begin the whole process over starting from the other side of the initial annular corridor A_0, but working in the opposite direction around z_1. A brief topological argument shows that we do not encounter the same difficulty again.

By the density condition on $E_{m,k}$, the diameter of J is bounded below by $C_3|z_1 - z_0|$ and we can find many pairs of points near z_n which are a given small distance $\sim C_4|z_1 - z_0|$ apart. Because the angles of the arcs $\gamma(\zeta, r)$ are larger than $\pi + \frac{1}{m}$ for a set of $r < r_1$ with measure larger than $C_5 r_1$ we can for each $\zeta \in J$ find a set of r's with measure larger than $C_6(m)r_1$ such that $\gamma(\zeta, r)$ intersects L_1 for each such r and so that these arcs continue past L_1 by an angle greater than $\alpha(m) > 0$. For fixed ζ, the set of intersections with L_1 has positive linear measure on L_1 larger than $C_7(m)r_1$. A pigeonhole argument on the different points in J then lets us find two pairs of arcs as shown in figure 9, passing through the same two points a and b on L_1, with $|a-b| \geq C_8|z_1 - z_0|$. The two centers of these pairs of arcs can be chosen far enough apart in J and the points a and b close enough together so that, as shown in the figure, the new annular corridor traverses an angle larger than $\alpha(m) > 0$ past L_1 with out meeting $\partial\Omega$. The newly constructed annular corridor has thickness greater than $C_9(m)|z_1 - z_0|$ and once it passes L_1 by an angle of $\alpha(m)$, we may do the S construction to make sure we keep things moving around z_1. Since $\alpha(m) > 0$ is fixed, we finish in finitely many steps, re-entering $D(z_0, |z_1 - z_0|)$ and obtaining our contradiction.

It only remains to start the construction. This is done by choosing $r_1 \sim |z_1 - z_0|$ so that $\partial D(z_0, |z_1 - z_0|)$ is close to being a diameter of $D(z_1, r_1)$. The density condition and the Beurling projection theorem provide many points of $E_{m,k}$ near z_1 so that we can construct the initial annular corridor A_0. There will be at most $\sim \frac{\pi}{\alpha(m)}$ steps in the construction and in each step the new annular corridor that is added may be dilated within a fixed factor compared to the previous one. This technicality is also easily dealt with by the initial choice for the size of A_0.

See [18] for the details.

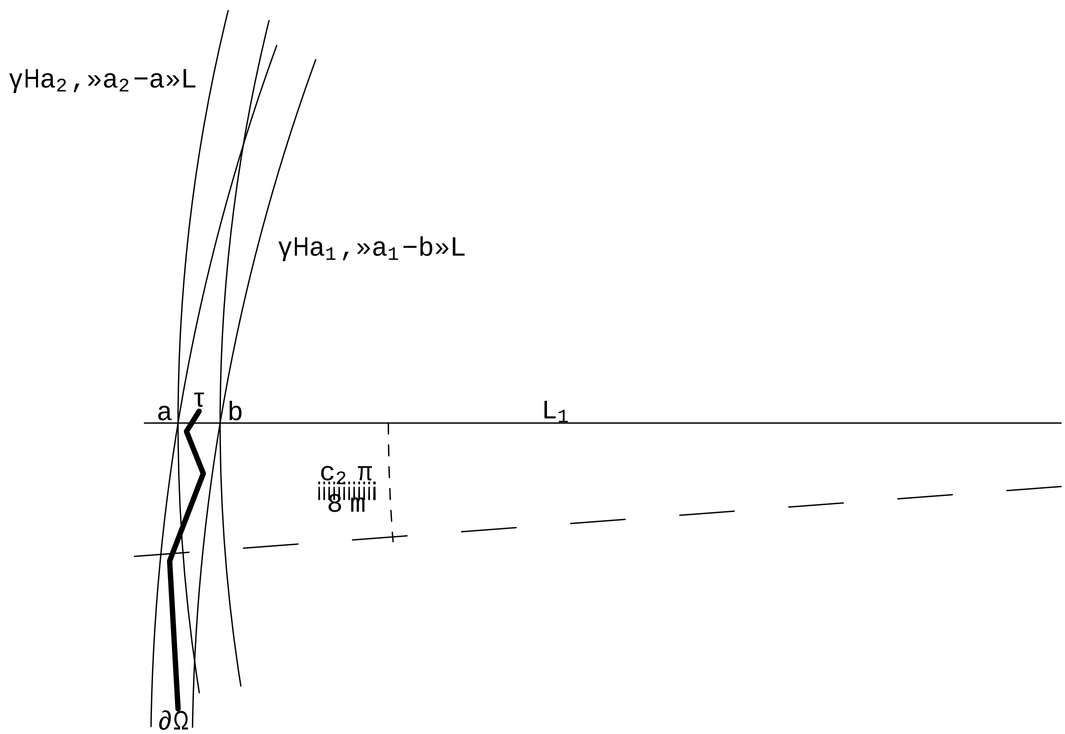

FIGURE 9. $S^*(a,b)$ can contain no point of $\partial\Omega$.

$\square$

7. Remarks and questions

1. Is there a proof of the twist point theorem which uses only geometry and Harnack's inequality? In [16], we will give a proof which does not use conformal mapping but requires Ito's formula and the harmonicity of the argument of the gradient of Green's function. The author was asked this question around 1994 by Juan Donaire and Jose Llorente.
2. With respect to Hausdorff dimension, the set of twist points which support harmonic measure is typically very small compared to the set of all twist points. Is there a purely geometric characterization of the twist points which support harmonic measure? What about if the

domain is of the type considered by Carleson in [7], i.e. if the harmonic measure can be represented as a shift invariant measure on a symbol space?

3. Do theorems 1,2 and 3 have natural generalizations to $\mathbb{R}^3$? Regarding theorem 1, K. Burdzy asked the following question (see [6] for similar questions).

   Let Ω be a domain in $\mathbb{R}^3$ and let X_t be a Brownian motion started at some fixed point $x_0 \in \Omega$ with η denoting a path. Let

$$\tau(\eta) = \inf\{t : X_t(\eta) \notin \Omega\}$$

   and

$$E(\eta) = \bigcap_{\epsilon>0} \overline{\left\{ \frac{X_t(\eta) - X_\tau(\eta)}{|X_t(\eta) - X_\tau(\eta)|} : \tau - \epsilon \leq t \leq \tau \right\}}.$$

   Is $E(\eta)$ almost surely either a sphere or a hemisphere? It is clearly a hemisphere if $\partial\Omega$ is smooth. If $\Omega = V \times \mathbb{R}$ where V is the Von-Koch snowflake domain then $E(\eta)$ is almost surely a sphere because almost every point of ∂V is a twist point.

4. The following conjecture is from [4]. Let $D(x,r)$ denote a disk of radius r centered at x. and let $\Omega \subset \mathbb{C}$ be simply connected. It is known from the work of Makarov, McMillan and Pommerenke, [11], [21], that for ω-a.e. $x \in \partial\Omega$

$$\lim_{r\to 0} \frac{\omega(D(x,r))}{r}$$

   exists and is not 0 or ∞. It is also known that for ω-a.e. twist point x,

$$\limsup_{r\to 0} \frac{\omega(D(x,r))}{r} = \infty.$$

   It is conjectured that

$$\liminf_{r\to 0} \frac{\omega(D(x,r))}{r} = 0$$

   for ω-a.e. twist point x.

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