MCMILLAN'S AREA PROBLEM.

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Abstract. We answer the question asked by McMillan in 1970 concerning distortion at the boundary by conformal mappings of the disk which was left open in our earlier paper [7].

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1. Introduction

Let A denote the set of ideal accessible boundary points of a simply connected domain Ω. Recall that these are the finite radial limit points of the Riemann map from the unit disk onto Ω and that each radius along which the limit exists gives a distinct ideal boundary point. In particular, distinct ideal accessible boundary points may have the same complex coordinate. Fix $w_0 \in \Omega$ and for each $a \in A$ and $r < |w_0 - a|$ let $\gamma(a, r) \subset \{z : |z - a| = r\}$ be the circular crosscut of Ω separating a from w_0 which can be joined to a by a Jordan arc contained in $\Omega \cap \{z : |z - a| < r\}$. Throughout this paper we will refer to $\gamma(a, r)$ as the principle separating arc for a of radius r.

Let $L(a, r)$ denote the Euclidean length of $\gamma(a, r)$ and let

$$A(a, r) = \int_0^r L(a, \rho)\, d\rho.$$

In [4], McMillan showed that

$$\limsup_{r \to 0} \frac{A(a, r)}{\pi r^2} \geq \frac{1}{2}$$

almost everywhere on $\partial\Omega$ with respect to harmonic measure (denoted by a.e.-ω hereafter).

The purpose of this paper is to prove

Theorem A.

$$\liminf_{r\to 0} \frac{A(a,r)}{\pi r^2} \le \frac{1}{2} \qquad a.e.\text{-}\omega$$

answering a question raised at the end of [4]. In an earlier paper, [7], we proved

Theorem B.

$$\liminf_{r\to 0} \frac{L(a,r)}{2\pi r} \le \frac{1}{2} \qquad a.e.\text{-}\omega$$

also in answer to the last paragraph of [4]. Theorem A implies Theorem B but the basic idea of the proof is the same as in [7]. Let

$$E_{m,k} = \{a \in A|\, A(a,r) > (\frac{1}{2} + \frac{1}{m})\pi r^2 \quad \forall r < \frac{1}{k}\}$$

and consider a Riemann map $f : \mathbb{D} \to \Omega$ from the unit disk to Ω such that $f(0) = w_0$. We will show that $f^{-1}(E_{m,k})$ has zero Lebesgue measure in the unit circle $\mathbb{T}$ for each m and k. We do this by showing that if $f^{-1}(E_{m,k})$ has a point of density for some m, k then the image of that point would be surrounded by a closed curve contained in Ω. Since the union of all such sets then has measure zero, this completes the proof.

The details of the present argument are more complicated than in [7] so it may be helpful to read [7] first to get the main idea with fewer technicalities. It may also be helpful to take an early glance at figures 1 and 2 near the end of the paper. For more detailed background on the problem, one can also refer to [4], [5] and [6]. For the ideas from geometric function theory used here we refer to [1], [8] and [3].

2. Proof of Theorem A.

In order to construct a curve in Ω which will surround a boundary point and thus give the contradiction which proves Theorem A, we will need to know that centered at almost every point of $E_{m,k}$ there is a wide angled annular corridor whose thickness is bounded from below. That such corridors exist will be a consequence of the accumulation of $E_{m,k}$ near the image of a point of density of $f^{-1}(E_{m,k})$. In fact, the abundance of points of $E_{m,k}$ will allow us to construct a chain of such corridors in Ω which will wrap around a boundary point.

We will require the following lemma. Let $\omega(z, E, \Omega)$ denote the harmonic measure of the set $E \subset \partial\Omega$ from the point $z \in \Omega$.

Lemma 2.1. *Let Ω be a simply connected domain in $\mathbb{C}$ and let f be a Riemann map $f : \mathbb{D} \to \Omega$. Let $E \subset \partial\Omega$ be a Borel set such that $f^{-1}(E)$ has a point of density. Then given $\delta > 0$ there is a point $w \in \Omega$ such that*

$$\omega(w, E, \partial\Omega) > 1 - \delta.$$

Proof. Let η be a point of density of $f^{-1}(E) \subset \mathbb{T}$. For any interval $I \subset \mathbb{T}$ centered at η there is a unique $r(I, \delta)$, $0 < r(I, \delta) < 1$ such that

$$\omega(r(I, \delta)\eta, I, \mathbb{D}) = 1 - \frac{\delta}{2}.$$

Let $z_I = r(I, \delta)\eta$. Given any $\epsilon > 0$ there is an interval I centered at η such that

$$|I \setminus f^{-1}(E)| < \epsilon |I|$$

where $|\cdot|$ denotes linear measure. Integrating the Poisson kernel at z_I over $I \setminus f^{-1}(E)$ then gives

$$\omega(z_I, I \setminus f^{-1}(E), \mathbb{D}) < \frac{\delta}{2}$$

if $|I|$ is sufficiently small. Therefore

$$\omega(z_I, I \cap f^{-1}(E), \mathbb{D}) > 1 - \delta$$

and taking $w = f(z_I)$ finishes the proof of the lemma. □

Let $d_f(z_I)$ denote the Euclidean distance from $f(z_I)$ to $\partial\Omega$. Actually, results of Beurling from [2] imply the existence of a constant K such that a disk of radius $Kd_f(z_I)$ contains all the harmonic measure of the set E found in the lemma. (See [8], pg.142.)

Let $w_0 = f(0)$ and assume that $\eta \in \mathbb{T}$ is a point of density of $f^{-1}(E_{m,k}) \subset \mathbb{T}$. The finite number of steps required to get a contradiction in the construction to follow will only depend on the number m in the definition of $E_{m,k}$. It will be clear from the construction that if $\delta > 0$ is sufficiently small and $\omega(w_1, E_{m,k}, \Omega) > 1 - \delta$ for some point w_1 then the required number of steps can be completed. Moreover, the choice of δ depends only on m. We choose δ to be this small and apply Lemma 2.1 with $E = E_{m,k}$ thus obtaining the desired point w_1.

Let d_0 be the Euclidean distance from w_1 to $\partial\Omega$ and let $x_0 \in \partial\Omega$ be a point such that $|x_0 - w_1| = d_0$. Since $f(\eta) \in A$ we can assume that $d_0 << \frac{1}{k}$ where k is the integer in the definition of $E_{m,k}$.

We will introduce positive constants $c_0, c_1, c_2, \ldots$ and $C_1, C_2, \ldots$. Their values will be determined in the discussion to follow and will either be purely numerical or depend only on m (in the definition of $E_{m,k}$). For any $w \in \mathbb{C}$ and $r > 0$, let $D(w, r)$ denote the set

$$\{z \in \mathbb{C} : |z - w| < r\}.$$

Let N be a large integer which will be determined later. We will see that it can be chosen so that $N \leq (\text{const. } m^{\frac{3}{2}})$. Since x_0 is a boundary point nearest to w_1 we may choose R_0 so that $D(w_1, d_0) \cap D(x_0, 2^N R_0)$ has area greater than $(\frac{1}{2} - \frac{1}{8m})\pi(2^N R_0)^2$. Choose c_0 so that if x_0^* is any point in $D(x_0, c_0 R_0)$ then the area of $D(w_1, d_0) \cap D(x_0^*, R_0)$ is greater than $(\frac{1}{2} - \frac{1}{4m})\pi R_0^2$. Later, we will also need $c_0 << \frac{1}{\sqrt{2m}}$. It is clear that R_0 is proportional to d_0 in a ratio depending only on m.

If $\delta > 0$ is sufficiently small, there is a set of points of $E_{m,k}$ of positive harmonic measure contained in $D(x_0, c_0 R_0)$. In fact, the circular arc $\partial D(x_0, c_0 R_0) \cap D(w_1, d_0)$ extends to a circular crosscut of Ω which determines a unique subdomain, U_0, of Ω not containing w_1. The midpoint, w^*, of the circular arc $\partial D(x_0, \frac{c_0 R_0}{2}) \cap D(w_1, d_0)$ is contained in U_0. By the comparison principle for harmonic measure and the Beurling projection theorem there is a constant $C_1 > 0$ such that

$$\omega(w^*, \partial U_0 \cap \partial\Omega \cap D(x_0, c_0 R_0), \Omega) \geq C_1 > 0$$

and by repeated application of Harnack's inequality in $D(w_1, d_0) \cup U_0$, there is then a constant C_2 such that

$$\omega(w_1, \partial U_0 \cap \partial\Omega \cap D(x_0, c_0 R_0), \Omega) \geq C_2 > 0.$$

By lemma 2.1, if δ is sufficiently small then

$$\omega(w_1, \partial U_0 \cap \partial\Omega \cap D(x_0, c_0 R_0) \cap E_{m,k}, \Omega) \geq \frac{C_2}{2} > 0, \tag{1}$$

as claimed.

Let x_0^* be an element of $\partial U_0 \cap D(x_0, c_0 R_0) \cap E_{m,k}$. Note that because $x_0^* \in E_{m,k}$ we have

$$\int_{\frac{R_0}{\sqrt{2m}}}^{R_0} L(x_0^*, \rho)\, d\rho \geq (\frac{1}{2} + \frac{1}{2m})\pi r^2$$

and by the choice of c_0, the area of

$$\{z \in \mathbb{C} : \frac{R_0}{\sqrt{2m}} \leq |z - x_0^*| \leq R_0\} \cap D(w_1, d_0)$$

is greater than $(\frac{1}{2} - \frac{1}{2m})\pi R_0^2$. If

$$\gamma(x_0^*, r) \cap D(w_1, d_0) = \emptyset$$

for each $r \in [\frac{R_0}{\sqrt{2m}}, R_0]$ then the area of the annulus

$$\{z \in \mathbb{C} : \frac{R_0}{\sqrt{2m}} \leq |z - x_0^*| \leq R_0\}$$

is greater than

$$(\frac{1}{2} + \frac{1}{2m})\pi R_0^2 + (\frac{1}{2} - \frac{1}{2m})\pi R_0^2 = \pi R_0^2.$$

This contradiction shows that there exists $r \in [\frac{R_0}{\sqrt{2m}}, R_0]$ such that $\gamma(x_0^*, r) \cap D(w_1, d_0) \neq \emptyset$. Simple topological considerations show that circular crosscuts of smaller radius centered at x_0^* which intersect $D(w_1, d_0)$ must be principle separating arcs for x_0^*. Let $c_1 = \frac{1}{\sqrt{2m}}$. Thus by shrinking R_0 by a factor no smaller than $\frac{c_1}{3}$, we may assume that for each $r \leq 3R_0$ we have $\gamma(x_0^*, r) \cap D(w_1, d_0) \neq \emptyset$. It follows that for each $r \leq 2R_0$ we have $\gamma(x_0, r) \cap D(w_1, d_0) \neq \emptyset$.

By a slight strengthening of the above argument it is clear that there are constants $c_2 > 0, c_3 > 0$ such that if $0 < R < \frac{1}{k}$ and $a \in E_{m,k}$ then

$$\Big|\{r \in [c_1 R, R] : L(a, r) > (1 + \frac{c_2}{m})\pi r\}\Big| \geq c_3 R. \tag{2}$$

We will now assume without loss of generality that x_0 is the origin and that w_1 is on the positive imaginary axis. Let

$$A_0 = \{z : R_0 < |z| < 2R_0\}$$

Let

$$\theta_0 = \inf\{\theta \in (-\frac{\pi}{2}, \pi) : J_\theta \cap \partial\Omega \neq \emptyset\}$$

where

$$J_\theta = \{z : \arg(z) = -\theta, R_0 < |z| < 2R_0\}.$$

Let

$$S_0 = \{z : R_0 < |z| < 2R_0, -\theta_0 \leq \arg(z) < \frac{\pi}{2}\}.$$

See figure 2.

Choose $x_1 \in J_{\theta_0} \cap \partial\Omega$. Let $R_1 = \frac{|x_1|}{2}$ and consider the annulus $A_1 = \{z : c_1 R_1 \leq |z - x_1| \leq R_1\}$. Any circular arc K centered at x_1 in A_1 with an angle of at least $(1 + \frac{c_2}{m})\pi$ is divided into two or three subarcs by the ray $\{z : \arg z = -\theta_0\}$. At least two of the arcs have an angle larger than $\frac{c_2 \pi}{2m}$. If $\alpha > 0$ is sufficiently small then the ray $L_1 = \{z : \arg z = -(\theta_0 + \alpha)\}$ also divides K into two or three subarcs, at least two of which have an angle larger than $\frac{c_2 \pi}{4m}$. The same angle α will be used in each step of the construction. It is determined so that in each new step, newly constructed annular corridors centered at points x_{j+1} with $\arg x_{j+1} = -\theta_j$ will cross the ray $\{z : \arg z = -(\theta_j + \alpha)\}$. The angle α does not depend on the size of R_1 (or R_j for later j) but only on c_1 and c_2. Specifically, choose $\alpha < \alpha^*$ where α^* is found by solving the triangle with sides $A = 1$, $B = \frac{c_1}{2}$ and C and angles $\angle AB = \pi - \frac{c_2\pi}{2m}$, $\angle CA = \alpha^*$ and $\angle BC$. The choice $\alpha = \frac{c_1 c_2 \pi}{32m}$ is sufficient for our purposes.

We can further choose a sufficiently small constant $c_4 > 0$ so that any circular arc centered at $a \in D(x_1, c_4 R_1)$ with an angle of at least $(1 + \frac{c_2}{m})\pi$ and with radius between $c_1 R_1$ and R_1 will also be divided by the ray L_1 into at least two subarcs with angle larger than $\frac{c_2\pi}{8m}$. Notice that c_4 depends only on c_1 and c_2 and not on R_1. We will use the same constant c_4 in subsequent similar steps of the construction with different radii R_j.

The circular arc $\partial D(x_1, c_4 R_0) \cap S_0$ extends to a crosscut of Ω which determines a subdomain U_1 not containing w_1. Because the width of S_0 is greater than $(const.)d_0$, we may argue as before using Harnack's inequality and the Beurling projection theorem in $D(w_1, d_0) \cup S_0 \cup U_1$ to find a constant $C_3 > 0$ depending only on m such that

$$\omega(w_1, \partial U_1 \cap \partial\Omega \cap D(x_1, c_4 R_0), \Omega) > C_3 > 0. \tag{3}$$

Therefore

$$\omega(w_1, \partial U_1 \cap \partial\Omega \cap D(x_1, c_4 R_0) \cap E_{m,k}, \Omega) > \frac{C_3}{2} > 0 \tag{4}$$

by lemma 2.1 with a sufficiently small initial choice of $\delta > 0$.

For each point $a \in E_{m,k} \cap \partial U_1 \cap D(x_1, c_4 R_0)$ let $F_a \subset [c_1 R_1, R_1]$ denote the set of r such that $L(a, r) > (1 + \frac{c_2}{m})\pi r$. By (2), the set F_a has $|F_a| > c_3 R_1$ and for each $r \in F_a$, $\gamma(a, r)$ intersects the ray L_1. Let x denote the orthogonal projection of x_1 on the line L_1. For points z, w in the plane, let $\overline{zw}$ denote the line segment with endpoints z and w. Then $L_1 = \overline{x_0 x} \cup \overline{x\{\infty\}}$ and we write $F_a = F_a^+ \cup F_a^-$ where F_a^+ (respectively, F_a^-) is the set of $r \in F_a$ such that $\overline{x\{\infty\}}$ (respectively, $\overline{x_0 x}$) divides $\gamma(a, r)$ into two subarcs, the smaller of which has an angle at least $\frac{c_2\pi}{8m}$. Then either $|F_a^+| \geq \frac{c_3}{2}R_1$ or $|F_a^-| \geq \frac{c_3}{2}R_1$. Making a choice of $+$ or $-$ so that the previous inequality holds, we rename the chosen set F_a^*. Let L_1^* denote the corresponding side of L_1 with respect to the point x and let

$$G_a = \{L_1^* \cap \gamma(a,r) : r \in F_a^*\}.$$

By (4) and the pigeonhole principal we find a_1 and a_1^* in $E_{m,k} \cap \partial U_1 \cap D(x_1, c_4R_0)$ and constants $c_5 > 0$ and $c_6 > 0$ so that $\frac{c_5R_0}{2} < |a_1 - a_1^*| < c_5R_0$ and so that $|G_{a_1} \cap G_{a_1^*}| > c_6R_1$. Note that here, $c_5 << c_4$. In fact it will be seen in the following paragraph that c_5 should be chosen to be small compared to the angle $\frac{c_2\pi}{8m}$.

There are now two cases to consider.

Case I. For each ρ such that $c_1R_1 \leq \rho \leq R_1$ we have $\gamma(a_1,\rho) \cap S_0 \neq \emptyset$.

Case II. There is some radius ρ with $c_1R_1 \leq \rho \leq R_1$, such that $\gamma(a_1,\rho) \cap S_0 = \emptyset$.

Assume that we are in Case I. Given a and b in $G_{a_1} \cap G_{a_1^*}$, let $S(a,b) \subset \Omega$ be the subdomain of Ω between the crosscuts $\gamma(a_1, |a_1 - a|)$ and $\gamma(a_1, |a_1 - b|)$. Let $S^*(a,b)$ denote the annular corridor bounded by $\gamma(a_1, |a_1 - a|)$, $\gamma(a_1, |a_1 - b|)$, $\overline{ab}$, and ∂S_0. We claim that there is a constant $c_7 > 0$ and there are points a and b in $G_{a_1} \cap G_{a_1^*}$ such that $|a - b| > c_7R_1$ and such that $S^*(a,b)$ contains no point of $\partial\Omega$. In fact, if $|a - b| < c_7^*R_1$ and if there is a point $\tau \in \partial\Omega$ contained in $S^*(a,b)$ then some piece of $\partial\Omega$ must connect τ to $\overline{ab}$ and then must extend past L_1 through an angle of at least $\frac{c_2\pi}{8m}$ in $S(a,b)$. Since c_5 is very small compared to $\frac{c_2\pi}{8m}$ and since $|a_1^* - a_1| \geq \frac{c_5R_0}{2}$, simple geometric considerations show that if c_7^* is sufficiently small, then one of the arcs $\gamma(a_1^*, |a_1^* - b|)$ or $\gamma(a_1^*, |a_1^* - a|)$ would intersect $\partial\Omega$ at a point too close to L_1 for the points a and b to be contained in $G_{a_1^*}$ (see Figure 1). As $|G_{a_1} \cap G_{a_1^*}| > c_6R_1$ and diam $(G_{a_1} \cap G_{a_1^*}) < (1 - c_1)R_1$ we find the desired constant c_7 with $c_7^* > c_7 > 0$ and the points a and b

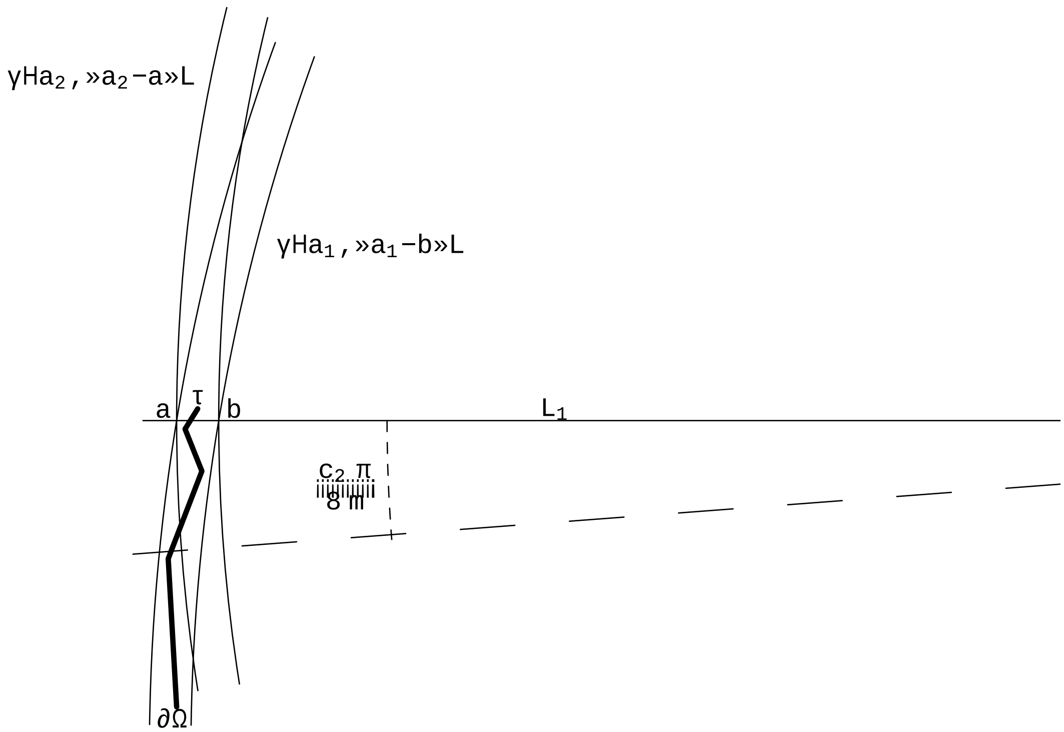

FIGURE 1. $S^*(a,b)$ can contain no point of $\partial\Omega$.

with $c_7 R_1 < |a-b| \leq c_7^* R_1$. Note that the constant c_7 only depends on previously introduced constants and therefore only on m. We rename the above annular corridor $S^*(a,b) \subset \Omega$ as S_0^*.

Now, still assuming Case I, let

$$J_\theta = \{z : \arg(z) = -\theta, \quad |a| < |z| < |b|\}$$

and let

$$S_1 = \{z : |a| < |z| < |b|, \quad -\theta_1 \leq \arg z \leq -(\theta_0 + \alpha)\}$$

where

$$\theta_1 = \inf\{\theta \in ((\theta_0+\alpha), \pi) : J_\theta \cap \partial\Omega \neq \emptyset\}.$$

See Figure 2.

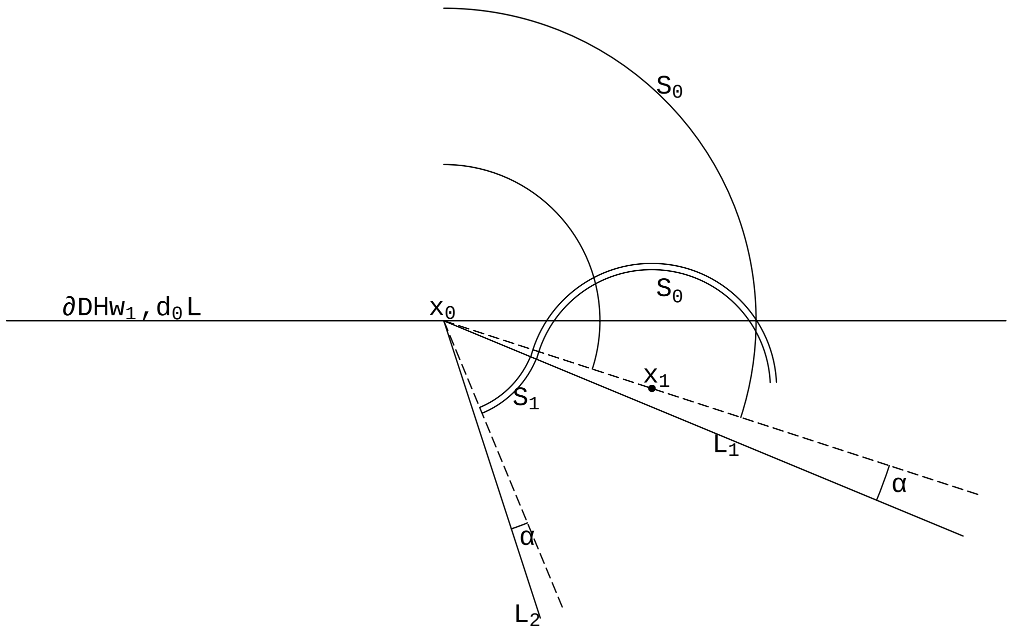

FIGURE 2. Step one of the construction.

Choose $x_2 \in J_{\theta_1} \cap \partial\Omega$. Let $R_2 = \frac{|x_2|}{2}$ and let L_2 be the ray $\{z : \arg z = -(\theta_1 + \alpha)\}$. The arc $\partial D(x_2, c_4 R_2) \cap S_0 \cup S_0^* \cup S_1$ defines a subdomain U_2 not containing w_1. Arguing as before with Harnack's inequality, the comparison principle and the Beurling projection theorem but now in $D(w_1, d_0) \cup S_0 \cup S_0^* \cup S_1 \cup U_2$ we find, using Lemma 2.1 with a sufficiently small choice of $\delta > 0$, a constant $C_4 > 0$ such that

$$\omega(w_1, \partial U_2 \cap \partial\Omega \cap D(x_2, c_4 R_2) \cap E_{m,k}, \Omega) \geq C_4 > 0.$$

As in the previous step we find points a_2 and a_2^* in $D(x_2, c_4 R_2) \cap \partial U_2 \cap E_{m,k}$ and sets $G_{a_2}, G_{a_2^*} \subset L_2$ with the same properties as before. We then have again

Case I. For each ρ such that $c_1 R_2 \leq \rho \leq R_2$ we have $\gamma(a_2, \rho) \cap S_0 \cup S_0^* \cup S_1 \neq \emptyset$.

and the complementary Case II.

Assume we are again in Case I. We repeat the argument made for the point x_1 at the new point x_2 and find two annular sectors. First S_1^* is found by the pigeon hole argument in the same way that S_0^* was found in the previous step. The new annular corridor S_1^* is centered at the point a_2 near x_2 and ends on the ray L_2 after having passed through the additional angle of α clockwise around x_0. Now S_2 is obtained in the same that S_1 was previously. That is, S_2 is centered at x_0, begins where S_1^* ends on L_2 and is stopped in its clockwise course around x_0 by a point $x_3 \in J_{\theta_2} \cap \partial\Omega$. In the j^{th} subsequent step a point x_j is found at the end of S_{j-1} and nearby points $a_j, a_j^* \in E_{m,k}$ are found as before. Case I at the j^{th} step means that every principle separating arc for a_j with radius ρ between $c_1 R_j$ and R_j intersects the union of the previously constructed annular corridors $S_0, S_0^*, S_1, S_1^*, \ldots, S_{j-1}$. The new annular corridors S_{j-1}^* and S_j are now found as in previous steps. Note that after the j^{th} step, the union of annular corridors so far constructed has turned through an angle of at least $j\alpha$ clockwise from the horizontal through x_0. A sufficiently small initial choice of $\delta > 0$ ensures that there is an abundance of points of $E_{m,k}$ near the point x_j at the end of S_{j-1} so that the construction may continue to the $(j+1)^{\text{st}}$ step.

Assuming that we only encounter Case I in each step, a sufficiently small choice of δ at the beginning of the proof allows us to repeat the argument $N = [\frac{2\pi}{\alpha}]$ times and this determines the choice of N at the beginning of the construction. Since the union of constructed corridors turns by an additional angle of at least α with each step, we will have constructed a connected union of annular corridors $\mathcal{C}$ in Ω contained in the annulus

$$\{z : 2^{-N} R_0 < |z - x_0| < 2^N R_0\}.$$

The union of $\mathcal{C}$ with $D(w_1, d_0)$ contains a closed curve in Ω surrounding the boundary point x_0.

If case II occurs at any step n before the N^{th} then there is a principle separating arc for a_n of radius ρ, $c_1 R_n \le \rho \le R_n$, which does not intersect

$S_0 \cup S_0^* \cup S_1 \cup S_1^* \cup \cdots \cup S_{n-1}$. It follows that the circular crosscut centered at a_n of radius ρ which does intersect $S_0 \cup S_0^* \cup S_1 \cup S_1^* \cup \cdots \cup S_{n-1}$ cannot be a separating arc for a_n at all. This means that w_0 is located in Ω on the concave side of this arc but on the convex side of the arcs which make up S_{n-1}. We then continue the construction at the $(n+1)^{\text{st}}$ step with the original annulus A_0 centered at x_0 but now turning in the counterclockwise direction. Since we have found case II in the clockwise direction, we cannot find case II in the counterclockwise direction without repeating the situation of w_0 being on the concave side of the last non-separating circular arc but on the convex side of the arcs in the last S_{n-1} from Case I. Simple topological considerations rule out this possibility and we therefore find a closed curve in Ω surrounding x_0 in at most N more steps.

It follows that there can be no point of density of $f^{-1}(E_{m,k})$ and that the harmonic measure of $E_{m,k}$ is therefore zero. The theorem is proved.

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