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A Green Proof of Fatou's Theorem

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Abstract

This paper is an exposition of some applications of Stochastic Processes to boundary behavior problems for harmonic functions. As an illustration, we give a proof of Fatou's theorem in simply connected plane domains which is probabilistic and does not use the Riemann mapping theorem. The paper closes with some remarks on further related work and open questions.

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1. Introduction

This paper comes from a talk given at a conference session on Applications of Stochastic Processes held in honor of M.M. Rao. The author thanks the organizers and Professor Rao for a very pleasant and informative meeting. The talk was a survey of topics and results centered on the connections between stochastic processes, potential theory and boundary behavior of harmonic functions. In the present paper we will focus on one point in the talk as an illustrative example of this relationship, presenting in detail a probabilistic and potential theoretic proof of a classical result of Fatou.

For $0 < r < 1$ and $-\pi \leq \theta \leq \pi$, let $P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos(\theta)}$ denote the Poisson kernel in the unit disk, and for $0 < \alpha < \pi$ let $\Gamma_\alpha(\theta)$ denote the convex hull of the point $\{e^{i\theta}\}$ and the circle of radius $\sin(\frac{\alpha}{2})$ centered at the origin. The region so defined has a vertex with angle α at $e^{i\theta}$. The following theorem is a result from Fatou's 1906 thesis and addresses the question of how solutions to the Dirichlet problem in the plane behave near the boundary of the domain.

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Theorem 1.1. *If $u(z)$ is a harmonic function in the unit disk with the representation*

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) F(t)\, dt, \qquad 0 \leq r \leq 1, \quad -\pi \leq \theta \leq \pi$$

for some $F \in L^p[-\pi,\pi]$ and $1 \leq p \leq \infty$, then for almost every $\theta \in [-\pi,\pi]$ with respect to Lebesgue measure, $u(z) \to F(\theta)$ as $z \to e^{i\theta}$ with $z \in \Gamma_\alpha(\theta)$.

In particular, $u(re^{i\theta})$ converges to $F(\theta)$ as $r \to 1$ for almost every θ. In order to state a possible generalization of this last statement to higher dimensions and more general domains, we require some terms whose definitions will be given precisely below. For now, readers less familiar with potential theory should compare the analogous parts of Theorem 1.1 where the domain Ω is the unit disk, the Green lines are radii and the measure $d\omega$ is given by integration against the Poisson kernel. A standard classical treatment of potential theory is given in Helms (1969). For more (or much more) on the connection with probability and stochastic processes see Port and Stone (1978), Bass (1995) or Doob (1984).

Let $\Omega \subset \mathbb{R}^n$ be a domain containing the origin and consider the family $\mathscr{P}$ of harmonic functions on Ω which have a representation of the form

$$u(x) = \int_{\partial\Omega} f(\xi)\, d\omega_x(\xi) \qquad x \in \Omega$$

where $d\omega_x$ denotes the harmonic measure in Ω and $f \in L^p(d\omega_0)$ with $1 \leq p \leq \infty$. As a consequence of Harnack's inequality (for which see for example Garnett and Marshall, 2005, p.27), the measures $d\omega_x$, $x \in \Omega$ are all mutually absolutely continuous, have the same sets of measure zero and give the same L^p spaces on $\partial\Omega$.

Say that Ω has property $\mathscr{F}$ if, for any $u \in \mathscr{P}$ we have

$$\lim_{x \to \xi, x \in \ell_\xi} u(x) = f(\xi), \qquad \xi \in \partial\Omega$$

for almost every trajectory ℓ_ξ of the Green function $g(0,\cdot)$ ending at $\xi \in \partial\Omega$. Here, the notion of almost every trajectory refers to the Green measure on trajectories which will be discussed below, but it is equivalent to consider the harmnonic measure of the set of endpoints $\xi \in \partial\Omega$.

It is not known whether general domains in $\mathbb{R}^n$ have property $\mathscr{F}$ or whether there is a good characterization of those that do. In $\mathbb{R}^2$, the fact that simply connected domains have property $\mathscr{F}$ follows from Theorem 1.1 by a conformal mapping argument. In $\mathbb{R}^n$ for $n \geq 3$, the absence of the conformal mapping technique makes results or conjectures which might appear to be straightforward extensions of two dimensional results, difficult or impossible to prove. In the following sections of the paper we will concentrate on the modest goal of proving that simply connected domains in the plane have property $\mathscr{F}$, without using a conformal mapping. Instead the proof will rely only on probabilistic and potential theoretic arguments. The new proof will give probabilistically intuitive reasons why the theorem is true and lay the groundwork for extending the ideas to higher dimensional generalizations. In the next section we will give background and some lemmas on harmonic measure, Green functions and other potential theoretic ideas, emphasizing the probabilistic viewpoint. In section 3 we give the proof that simply connected domains in $\mathbb{R}^2$ have property $\mathscr{F}$. In section 4 we close with some remarks on further work and some related open questions.

2. Brownian Motion and Harmonic Measure

The term Harmonic Measure (Harmonische Mass) originates in the monograph (Nevanlinna, 1944) of R. Nevanlinna as the name for a useful conformally invariant quantity in problems of geometric function theory. The technique had already been in use by (e.g.) Carleman, Lindelöf, and both F. and R. Nevanlinna. We will jump ahead a bit and take as our starting point the potential theoretic description of harmonic measure in terms of the Perron-Wiener-Brelot solution of the Dirichlet problem.

A real valued function u defined on a domain Ω in $\mathbb{R}^n$ is superharmonic if it is lower semi-continuous on Ω, not identically infinite on any component of Ω and if for any $x \in \Omega$ the average value of u on small balls centered at x does not exceed $u(x)$. A function u defined on Ω is subharmonic if $-u$ is superharmonic.

Let Ω be any domain in $\mathbb{R}^n$ and let f be a function defined on $\partial\Omega$. The upper class of functions for f, denoted by U_f, includes the function which is identically $+\infty$ on Ω and otherwise consists of all superharmonic functions u defined in Ω which are bounded below and satisfy, for each $Q \in \partial\Omega$,

$$\liminf_{\Omega \ni x \to Q} u(x) \geq f(Q).$$

The lower class L_f of functions for f has the mirror image definition in terms of an upper envelope of subharmonic functions or equivalently

$$L_f = -U_{(-f)}.$$

The upper and lower solutions of the generalized Dirichlet problem for f are respectively,

$$\overline{Hf}(x) = \inf\{u(x) : u \in U_f\}$$

and

$$\underline{Hf}(x) = \sup\{u(x) : u \in L_f\}.$$

If $\overline{Hf}(x)$ and $\underline{Hf}(x)$ are equal and harmonic in Ω, we write $Hf(x) = \overline{Hf}(x) = \underline{Hf}(x)$ and say that f is a resolutive boundary function. It is a result of Wiener (1924) that each continuous real function on $\partial\Omega$ is resolutive. It can be shown further that the space of bounded resolutive functions is a Banach space in the supremum norm on $\partial\Omega$ and that the operator H is linear on this space. By the maximum principle and the Riesz representation theorem, there is for each $x \in \Omega$, a unique Borel measure ω_x on $\partial\Omega$ such that for each $f \in C(\partial\Omega)$,

$$Hf(x) = \int_{\partial\Omega} f \, d\omega_x.$$

The measure ω_x is called the harmonic measure of Ω evaluated at x. The harmonic measure can be considered to be a function of three arguments; Ω the domain, $E \subset \partial\Omega$ the set being measured and $z \in \Omega$, the base point for the measure. We will suppress the arguments of the harmonic measure when it is convenient and not confusing. When necessary for clarity we will expand the notation completely, writing $\omega(z,E,\Omega)$ for the harmonic masure at $z\in\Omega$ of $E\subset\partial\Omega$ in the domain Ω. When it is more convenient typographically, we will write this as $\omega_z(E,\Omega)$ or just $\omega_z(E)$ when Ω is understood.

It was Kakutani who first made the connection between Brownian Motion and harmonic measure in Kakutani (1944), by showing that the harmonic measure ω_x is the exit distribution of a Brownian motion in Ω started at x. With Ω as before, let $E\subset\partial\Omega$ be a Borel set and fix $a\in\Omega$. Let X_t denote a Brownian motion started at a and let

$$\tau=\inf\{t>0 : X_t\notin\Omega\}.$$

Then

$$\omega_a(E)=\mathbb{P}^a\{X_\tau\in E\}.$$

In the seminal paper (Doob, 1954), Doob extends the ideas of Kakutani and provides complete details for a more general treatment. In the process, he lays out a complete theory connecting (super)martingales and (super)harmonic functions. A martingale is a model of a fair game, and with the ideas of Kakutani and Doob we may interpret the Dirichlet problem with resolutive boundary data f as just such a game. A player located at $x\in\Omega$ randomly chooses a Brownian path according to the Wiener distribution and receives a payoff equal to the value of f at the first exit point of the path from Ω. The player's expected payoff is a function $u(x)$ which is harmonic and which solves the Dirichlet problem. Doob also shows, (Doob, 1954, Theorem 6.2), the following fact which we will think of as

Theorem 2.1 (Doob's probabilistic version of Fatou's theorem)**.** *If u solves the Dirichlet problem on Ω with resolutive boundary data f, then*

$$\lim_{t\uparrow\tau} u(X_t)=f(X_\tau). \qquad a.s.\ \mathbb{P}^x$$

where X_t is a Brownian motion started at x and $\mathbb{P}^x$ denotes Wiener's measure on paths started at x.

Note that Theorem 2.1 provides, in one sense, a far reaching generalization of Fatou's theorem since it gives a boundary convergence result in any domain and in any dimension. In another sense, the hypotheses of Theorem 2.1 are much weaker though since convergence is allowed to occur along almost any randomly chosen Brownian path exiting the domain.

A few years after Doob's paper (Doob, 1954) appeared, the works of Hunt, (Hunt, 1957a,b, 1958) (and other probabilists such as Kac, Kemeny and Snell, Spitzer and Kesten) extended the new probabilistic potential theory from Brownian motion to the setting of general homogeneous Markov processes. These works created a dictionary describing the non-trivial analogies between potential theoretic and probabilistic notions. As an example from this dictionary we consider the potential theoretic Green function and the Green function defined in the theory of random walk on a square lattice. Given a domain $\Omega\subset\mathbb{C}$ and $z_0\in\Omega$, let

$$u(z)=\int_{\partial\Omega}\log|z_0-\xi|\,\omega_z(d\xi)$$

be the solution to the Dirichlet problem with boundary data $\log|z_0-\xi|$. The Green function

$g(z_0,z)$ is defined to be

$$g(z_0,z) = \log \frac{1}{|z-z_0|} + u(z)$$

so that g is harmonic in $\Omega \setminus \{z_0\}$, g tends to zero at the boundary, and g has a logarithmic singularity at z_0. One shows that g is uniquely defined and that for smooth $\partial\Omega$,

$$d\omega_{z_0} = \frac{\partial g}{\partial n}(z_0,\cdot)|d\xi|$$

where n denotes the unit outward normal vector and $|d\xi|$ is the arclength on $\partial\Omega$.

In the theory of random walk on a lattice (say $\mathbb{Z}^2$), for a bounded set A of lattice points, we define the Green function $g_A(x,y)$ for $A^c \times A^c$ to be the expected number of visits to y of a random walk starting at x before hitting A. Lemma 2.8 below gives an example of the analogy between these two Green functions.

An alternative definition of harmonic measure can be given in terms of the trajectories of the gradient of Green's function, and this was done by Brelot and Choquet (1951). Note that the previous definitions become vacuous in case Ω has no Green function and assume from now on that all domains considered possess one. By the work of Doob, we know that this means exactly that Ω^c has positive probability of being visited by a Brownian traveler starting at any $z \in \Omega$. To briefly summarize some of the results of Brelot and Choquet, let $a \in \Omega$ be fixed and let $g_a(x)$ be Green's function for Ω with pole at a. The Green lines starting at a are the maximal orthogonal trajectories of the level lines of g_a which have a limit point at a. Each Green line has a well defined initial direction given by its unit tangent vector at a and each point on the unit sphere corresponds in this way to a Green line at a. Given a Borel subset $E \subset \partial\Omega$, the Green's measure of E is defined to be the normalized Lesbesgue measure of the set of unit tangent vectors on the sphere for which the corresponding Green lines terminate at points of E. Denote the Green's measure of E by $G_a(E)$ and take the normalization so that $G_a(\partial\Omega) = 1$. With respect to Green's measure, almost every Green line terminates in a point of $\partial\Omega$ and such Green lines are called regular. Let $\overline{G_a}$ and $\underline{G_a}$ denote respectively the outer and inner Green measures and let $\overline{\omega_a}$ and $\underline{\omega_a}$ denote the outer and inner harmonic measures for general subsets of $\partial\Omega$. Brelot and Choquet proved that for any subset $A \subset \partial\Omega$

$$\overline{\omega_a}(A) \le \underline{G_a}(A) \le \overline{G_a}(A) \le \overline{\omega_a}(A) \tag{2.1}$$

so that harmonic measurability implies Green measurability with equality of the measures. Later, Arsove (1972), proved the converse of this statement, thereby showing that the two measures are the same.

We will not go into the details of the arguments in Brelot and Choquet (1951) but we remark that the main idea behind them is to carefully apply Green's theorem. One defines a tube as the set of Green lines connecting neighborhoods on disjoint level surfaces of g_a and uses Green's formula to show that the flux of the vector field of ∇g_a is the same at each end of the tube. Letting one end of a tube tend to the boundary and the other to the singularity of g_a we get the equivalence of the normalized Lebesgue measure on a sphere centered at a for one end of the tube with the harmonic measure of the endpoints of the Green lines on $\partial\Omega$ at the other. Difficulties posed by critical points are circumvented by using the fact that the critical set of g_a corresponds to a set of measure zero in the normalized spherical measure at a. Covering subsets of the boundary by ends of tubes and attending to the details leads to (2.1).

We will require the following basic projection estimates of harmonic measure due to Beurling and Hall. The original proofs are in Beurling (1933) and Hall (1937) respectively. Here we will give the statements as they appear in Øksendal (1983) where stochastic proofs using the strong Markov and reflection properties of Brownian motion are given. Versions of the results for $\mathbb{R}^3$ are also given in Øksendal (1983).

Let $0 \leq R_1 < R_2 \leq \infty$ and let A denote the annulus

$$A = \{z : R_1 \leq |z| \leq R_2\}$$

Let $K \subset A$ be compact and let $-R_2 < a < -R_1$. Put

$$K^* = \{|z| : z \in K\} \subset \mathbb{R} \subset \mathbb{C}$$

and define $U = A^\circ \setminus K$ and $V = A^\circ \setminus K^*$.

Lemma 2.1 (The Beurling projection theorem)**.**

$$\omega_a(K,U) \geq \omega_a(K^*,V).$$

With the same notations, suppose that

$$R_1 < r_1 < r_2 < R_2$$

Lemma 2.2 (Hall's Lemma)**.** *There exists $c > 0$ such that for all compact $K \subset \{z : r_1 < |z| < r_2\}$,*

$$\omega_a(K,M) \geq cm_1(K^*)$$

where $M = A^\circ \setminus K \setminus [0,\infty)$ and m_1 denotes one dimensional Lebesgue measure on $\mathbb{R}$.

The probabilistic content of the projection theorems is perhaps intuitively clear. Among all sets with the same projections, the ones which are best hidden from a Brownian motion started at a are prescribed by the theorems.

We can combine Hall's lemma and the strong Markov property to get the following well known lemma in the plane simply connected case. We require a standard notation for disks shall use, here and elsewhere, $D(z,r)$ for the open Euclidean disk with center z and radius r.

Lemma 2.3. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Given $\varepsilon > 0$ there is $c_0 > 0$ such that for any $z \in \Omega$*

$$\omega(z, \partial\Omega \cap D(z, c_0 \mathrm{dist}(z,\partial\Omega)), \Omega) > 1 - \varepsilon$$

where dist *denotes the Euclidean distance in the plane.*

Proof. The proof may be accomplished by a standard application of Hall's lemma, see Øksendal (1983), and the strong Markov property. Similar arguments have been used in many places, see for example the survey article Betsakos (2001) and the papers referred to there.

Let $r = \mathrm{dist}(z, \partial\Omega)$, $D_k = \{w : |w-z| \le 2^k r\}$, $A_{k+1} = D_{k+1} \setminus D_k$ for $k = 0,1,2,\ldots$. Then

$$\begin{aligned}\omega(z, \partial\Omega \cap D_k^c, \Omega) &= \int\limits_{\partial D_{k-1}\cap\Omega} \omega(w, \partial\Omega \cap D_k^c, \Omega)\, \omega(z, dw, \Omega \cap D_{k-1}) \\ &\le \max_{w\in\partial D_{k-1}\cap\Omega} \omega(w, \partial\Omega\cap D_k^c, \Omega)\omega(z, \partial D_{k-1}\cap\Omega, \Omega\cap D_{k-1})\end{aligned}$$

and for $j < k$ we have in the same way

$$\omega(z, \partial D_j \cap \Omega, \Omega \cap D_j) = \int\limits_{\partial D_{j-1}\cap\Omega} \omega(w, \partial D_j \cap \Omega, \Omega \cap D_j)\, \omega(z, dw, \Omega \cap D_{j-1})$$

$$\le \max_{w\in\partial D_{j-1}\cap\Omega} \omega(w, \partial D_j \cap \Omega, \Omega \cap D_j)\omega(z, \partial D_{j-1} \cap \Omega, \Omega \cap D_{j-1}).$$

If there is $w \in \partial D_{j-1} \cap \Omega$ then since Ω is simply connected, the circulular projection of $\partial\Omega$ onto a radius of A_{j-1} is an onto mapping. An application of Hall's lemma in the annulus $A_j \cup A_{j-1}$ then shows that there is a universal constant $0 < c_1 < 1$ such that

$$\max_{w\in\partial D_{j-1}\cap\Omega} \omega(w, \partial D_j \cap \Omega, \Omega \cap D_j) \le (1 - c_1).$$

(Note that the conclusion of Hall's lemma is scale invariant.) It then follows by induction that $\omega(z, \partial\Omega \cap D_k^c, \Omega) \le C(1-c_1)^k$. Taking k sufficiently large so that $C(1-c_1)^k < \varepsilon$ proves the lemma with $K \ge 2^k$. □

The next lemma and the definitions that precede it relate the exits of Brownian travelers to the Green line definition of harmonic measure.

Given a domain $\Omega \subset \mathbb{R}^n$ and a fixed point $x_0 \in \Omega$, choose an angle $0 < \alpha < \pi$ and consider a regular Green line $\ell(x_0, x_1)$ starting from x_0 and passing through some other point $x_1 \in \Omega$. Let v be the unit tangent vector to $\ell(x_0, x_1)$ at x_1 pointing in the direction of decreasing $g(x_0, \cdot)$ along $\ell(x_0, x_1)$. We define the forward cone $\Lambda_\alpha(x_1)$ to be the set of Green lines starting at x_1 whose tangent vectors at x_1 are at an angle less than α with v.

In a plane domain Ω, define forward cones with respect to a fixed base point z_0. As a replacement for the regions $\Gamma_\alpha(e^{i\theta})$ used in Fatou's theorem we have

Definition 2.1. *The Green cone of aperture α over $\zeta \in \partial\Omega$ is denoted $\Gamma_\alpha(\zeta)$ and defined as*

$$\Gamma_\alpha(\zeta) = \{z \in \Omega : \zeta \in \partial\Lambda_\alpha(z)\}.$$

The geometry of Green cones is not completely out of control.

Lemma 2.4. *For any $\lambda > 0$ and any two points p_1 and p_2 in*

$$\Gamma_\alpha(\zeta) \cap \{z : g(z_0, z) = \lambda\}$$

there is an N depending only on α and a sequence of $m \le N$ disks $D_i \subset \Omega, 1 \le i \le m$ such that D_1 is centered at p_1, D_m is centered at p_2, consecutive disks intersect and each disk has radius comparable to its distance to $\partial\Omega$.

The proof of this fact is omitted here since it relies only on facts about Green lines and Harnack's inequality. Lemma 2.4 is proved without using a conformal mapping in O'Neill (Preprint). Notice that it implies that if $u \geq 0$ is harmonic in Ω then the values of u on $\Gamma_\alpha(\zeta) \cap \{z : g(z_0, z) = \lambda\}$ are all comparable. That is, there are universal constants C_1 and C_2 such that $C_1 u(w_1) \leq u(w_2) \leq C_2 u(w_1)$ for any $w_1, w_2 \in \Gamma_\alpha(\zeta) \cap \{z : g(z_0, z) = \lambda\}$.

In what follows, we will identify Green lines starting from some fixed base point $z_0 \in \Omega$, with ideal boundary points of Ω. By the Moore triod theorem of plane topology, the number of points in the Euclidean boundary, $\partial\Omega$, which are the endpoints of more than two regular Green lines is at most countable. In many situations, it is possible to reduce matters to the case in which there is exactly one Green line ending at each Euclidean boundary point by considering interior approximation by Jordan domains.

Let $X_t(\eta)$ denote Brownian motion started at the fixed base point $X_0(\eta) = z_0$ and let $\tau(\eta) = \inf_t \{X_t(\eta) \notin \Omega\}$ denote the first exit time from Ω. Let K be a compact subset of $\partial\Omega$ and let $C(K, \alpha) = \bigcup_{\zeta \in K} \Gamma_\alpha(\zeta)$. With these notations, we have the following lemma.

Lemma 2.5. *For almost every Brownian path η such that $\tau(\eta) \in K$, there is $t_0(\eta) < \tau(\eta)$ such that*

$$X_t(\eta) \in C(K, \alpha), \qquad t_0 \leq t < \tau.$$

Proof. Given $z \in \Omega$ and $\alpha \in (0, \pi)$ we may choose $\lambda > 0$ such that if $w \in \{w : g(w, z) = \lambda\} \cap \Lambda_{\frac{\alpha}{2}}(z)$ then $\omega(w, \partial\Lambda_\alpha \cap \partial\Omega, \Lambda_\alpha^\circ) \geq \frac{1}{2}$. With such a choice of λ define a sub-domain $U(z, \alpha) \subset \Omega$ as

$$U(z, \alpha) = \{w \in \Omega : g(w, z) > \lambda\} \cup \Lambda_\alpha(z).$$

Then we have $\omega(z, \partial\Lambda_\alpha(z) \cap \partial\Omega, U(z, \alpha)) \geq \frac{\alpha}{8\pi}$. Now define stopping times as follows:

$$\tau_1(\eta) = \inf\{t > 0 : X_t(\eta) \notin C(K, \alpha)\},$$

$$\tau_2(\eta) = \inf\{t > \tau_1 : X_t(\eta) \in C\left(K, \frac{\alpha}{2}\right) \setminus U(X_{\tau_1}(\eta), \alpha)\},$$

$$\tau_{2k+1}(\eta) = \inf\{t > \tau_{2k} : X_t(\eta) \notin C(K, \alpha)\},$$

and

$$\tau_{2k}(\eta) = \inf\{t > \tau_{2k-1} : X_t(\eta) \in C\left(K, \frac{\alpha}{2}\right) \setminus U(X_{\tau_{2k-1}}(\eta), \alpha)\}.$$

We have, from the definitions of $C(K, \alpha)$, $U(z, \alpha)$, and from the strong Markov property that

$$P\{\tau_{2k} > \tau | X_\tau \in K\} = 0,$$

$$P\{\tau_{2k} < \tau | \tau_{2k-1} < \tau\} < 1 - \frac{\alpha}{8\pi}$$

and

$$P\{X_\tau \in K | \tau_{2k+1} < \tau\} < 1 - \frac{\alpha}{8\pi}.$$

It follows that

$$P[\{X_\tau \in K\} \cap \{\tau_{2k+1} < \tau\}] < \left(1 - \frac{\alpha}{8\pi}\right) P\{\tau_{2k+1} < \tau\}$$
$$\leq \left(1 - \frac{\alpha}{8\pi}\right) P\{\tau_{2k} < \tau\}$$

and that

$$P\{\tau_{2k} < \tau\} = P[\{\tau_{2k} < \tau\} \cap \{\tau_{2k-1} < \tau\}] \leq \left(1 - \frac{\alpha}{8\pi}\right) P\{\tau_{2k-1} < \tau\}.$$

By induction, we then have

$$P[\{X_\tau \in K\} \cap \{\tau_{2k+1} < \tau\}] < (1 - \frac{\alpha}{8\pi})^k.$$

By the Borell-Cantelli lemma, with probability one only finitely many of the events $\{X_\tau \in K\} \cap \{\tau_{2k+1} < \tau\}$ occur. The statement of the lemma follows. $\square$

The following three lemmas are technical results about harmonic measure whose potential theoretic proofs may be lifted from the paper Jerison and Kenig (1982). The statements have been altered to suit our purposes here. We include the proof of the first one since it is short. For any point $w \in \Omega$ let $d_\Omega(w)$ denote the Euclidean distance from w to $\partial\Omega$. We will use this notation several times below.

Let $r_0 > 0$ be small compared to $d_\Omega(z_0)$, say $r_0 < 2^{-10} d_\Omega(z_0)$, and let $\xi \in \partial\Omega$ be fixed. Suppose that for each $k = 0, 1, 2, \ldots$, s_k is an arc of a circle centered at ξ with radius $2^{-k} r_0$. Suppose also that each s_k separates the same ideal boundary point at ξ from z_0 in Ω. Let u be a positive harmonic function in Ω that vanishes continuously on γ, the subset of the ideal boundary of Ω which is separated from z_0 by s_0. With this setup and notation we have the following lemma.

Lemma 2.6. *There is $\beta > 0$ which does not depend on Ω such that*

$$u(z) \leq 2^{-\beta k} (\sup_{s_0} u) \text{ for } z \in s_k \text{ and } k = 0, 1, 2, \ldots$$

Proof. We shall use the method given in Jerison and Kenig (1982) Lemma 4.1.

Let $M = \sup_{s_0} u$ and let v be harmonic in the component Ω' of $\Omega \setminus s_0$ not containing z_0, with $v = 1$ on s_0 and $v = 0$ on γ. Then $u \leq Mv$ on Ω' by the maximum principle, so we show that $v \leq 2^{-\beta k}$ on s_k.

On s_1, $v \leq 1 - \varepsilon_0$ for some $\varepsilon_0 > 0$ by the Beurling projection theorem. Let $\beta > 0$ be such that $1 - \varepsilon_0 = 2^{-\beta}$. In the same way, for $k \geq 1$, if $v \leq 2^{-\beta k}$ on s_k, then $2^{\beta k} v \leq 2^{-\beta}$ on s_{k+1}. $\square$

The next lemma provides local control over harmonic functions which vanish continuously at the boundary.

Suppose a circular arc s_{2r} of radius $2r$ is centered at $\xi \in \partial\Omega$ and separates an ideal boundary point at ξ from z_0. Let γ denote the part of the ideal boundary separated from z_0 by s_{2r}. Let s_r be an arc of radius r centered at ξ which separates the same ideal boundary point from z_0 and let $z_r \in s_r \cap \Omega$. Suppose that u is harmonic in Ω, $u \geq 0$ and vanishes continuously on γ. Let Ω_r denote the component of $\Omega \setminus s_r$ which does not contain z_0.

Lemma 2.7. *There is $C > 0$ depending on $\frac{d_\Omega(z_r)}{r}$ but not depending on Ω or on u such that*

$$u(z) \leq Cu(z_r)$$

for all $z \in \Omega_r$.

Proof. The proof follows the reasoning in Jerison and Kenig (1982), lemma 4.4, which has its origin in Carleson (1962). It can also be considered as a nice exercise in the use of the previous lemma. □

For the last lemma we control the local geometry. Assume that Ω is simply connected and that $\partial\Omega$ contains a line segment L. Suppose that $\xi \in L$ and $r > 0$ are such that a diameter L' of the disk $D(\xi, r)$ is contained in L and that L' divides $D(\xi, r)$ into a half contained in Ω and a half contained in Ω^c. Suppose that $z_0 \in \Omega \setminus D(\xi, r)$. Let $z_{r/2}(\xi) \in \partial D(\xi, r/2) \cap \Omega$ be the point with Euclidean distance $r/2$ to L'. Under these assumptions we have the following.

Lemma 2.8. *There are constants C_1 and C_2 which do not depend on Ω or z_0 or on $r > 0$ such that for $\xi \in L$,*

$$C_1 g(z_0, z_{r/2}(\xi)) \leq \omega_{z_0}(D(\xi, r/2) \cap \partial\Omega) \leq C_2 g(z_0, z_{r/2}(\xi))$$

The proof is again omitted but we will make a few remarks on the origins of the main ideas. For bounded Ω, the argument follows that of the proof of inequalities 4.3 and 4.6 in Jerison and Kenig (1982). The Beurling projection theorem replaces the existence of exterior non-tangential balls. In the unbounded case, the replacement of the Newtonian potential by the logarithmic potential requires an estimate of

$$\int_{\partial\Omega_{\text{far}}} \log|\xi - w|\, d\omega_{z_0}(w)$$

where the integral is over the part of $\partial\Omega$ far from ξ (as compared to $|z_0 - \xi|$). This may be done by an application of Hall's lemma as in the proof of lemma 2.3. The reasoning given in Jerison and Kenig (1982) has its origin in Dahlberg (1977). Note the analogy between the two dimensional potential theoretic Green function and the Green function for a discrete random walk. We may compare $g(z_0, z_{r/2}(\xi))$ to the expected number of crossings by a Brownian motion started at z_0 of the annulus centered at $z_{r/2}(\xi)$ with radii $r/4$ and $r/8$, before exiting Ω. This expectation is in turn comparable to the probability that the same Brownian motion visits the interior disk of the annulus.

3. Fatou's theorem on Green lines

We now turn to a proof of the generalization of Theorem 1.1 to simply connected domains in the plane. For the sake of simplicity we will state the theorem here for Jordan domains (so that $\partial\Omega$ is homeomorphic to a circle) thus avoiding a more complicated statement involving ideal boundary points or Martin boundary points and the introduction of the Martin kernel in place of $\omega_z(d\xi)$. The proof below is easily adapted to the general case by considering the Green measure as a measure on sets of Green lines with each regular Green line ℓ_ξ corresponding to an ideal boundary point ξ.

Theorem 3.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected Jordan domain and let $z_0 \in \Omega$ be fixed. Let*

$$u(z) = \int_{\partial\Omega} f(\xi)\, \omega_z(d\xi)$$

where ω_z denotes the harmonic measure at $z \in \Omega$ and $f \in L^p(d\omega_{z_0})$ for some $1 \le p \le \infty$. Let E denote the set of all Green lines ℓ_ξ with $\xi \in \partial\Omega$ such that $\lim_{z\to\xi, z\in\ell_\xi} u(z) = f(\xi)$. Then $\omega_{z_0}(E,\Omega) = 1$.

Recall that, in its general form, this theorem is the conformally invariant version of Theorem 1.1 and that our goal is to prove it without using the Rieman mapping theorem. We will first outline a separate argument for the case where the boundary data f is in $L^\infty(d\omega)$. This argument emphasizes the probabilistic intuition underlying the theorem. For the proof in the general situation of $f \in L^p$, $1 \le p \le \infty$, we will use a classical maximal function argument, but one which still relies on the strong Markov property.

The basic building block for our proof is the construction of a “cap". As before, let z_0 denote a fixed point in Ω. We will construct a cap at the point $z_1 \in \Omega$ as follows.

Let ℓ denote the Green line starting at z_0 which passes through z_1. Almost all $\xi \in \partial\Omega$ with respect to harmonic measure are the endpoints of Green lines, and we assume ℓ has the endpoint ξ. At z_1 there is a tangent vector v to ℓ pointing toward ξ along ℓ. Let $0 < \alpha < \pi$ and consider the pair of Green lines ℓ_+ and ℓ_- which start at z_1 and whose tangents at z_1 make an angle of $\pm\alpha$ with v respectively. The total angle subtended by ℓ_+ and ℓ_- is 2α, so the Green lines between them sweep out a subset E of the boundary with $\omega_{z_1}(E) = \frac{\alpha}{\pi}$. Denote the endpoints on the boundary of $\ell_\pm$ respectively by $\xi_\pm$. Let $\delta > 0$, to be chosen more precisely later, and follow ℓ_+ and ℓ_- away from z_1 until the first points $z_+ \in \ell_+$ and $z_- \in \ell_-$ such that

$$d_\Omega(z_\pm) < \delta d_\Omega(z_1).$$

We now re-use the notation and let ℓ_+ and ℓ_- denote the segments of the Green lines joining z_1 to $z_\pm$. Let $p_\pm \in \partial\Omega$ be a nearest boundary point to $z_\pm$ and let $L_\pm$ denote a straight line segment connecting $z_\pm$ and $p_\pm$.

Definition 3.1. *With the notations given above the cap at z_1 over the point $\xi \in \partial\Omega$ is the union*

$$cap(z_1,\xi) = L_+ \cup \ell_+ \cup \ell_- \cup L_-.$$

By a simple limiting argument, every point of $\partial\Omega$ which is the vertex of a triangle contained in Ω is the endpoint of a regular Green line starting at z_0, so the points $p_\pm$ correspond to Green lines from z_0.

With this setup we have the following

Lemma 3.1. *The $cap(z_1,\xi)$ separates ξ from z_0 in the sense that any Jordan arc contained in Ω which joins z_0 to ξ must intersect $cap(z_1,\xi)$.*

Proof. To see this, choose, at z_+, Green lines γ_1^+, γ_2^+ so that γ_1^+ makes a positive angle with the tangent in the direction toward ξ_+ to ℓ_+ at z_+ and so that similarly γ_1^- makes a negative angle. By lemma 2.3, we may choose γ_1^+, γ_1^- so that they terminate at boundary points whose distance to z_+ is no larger than $K\delta d_\Omega(z_1)$. Here $\delta > 0$ is from the choice of z^+

and K is the constant from lemma 2.3. The subset of the ideal boundary separated from z_1 by $L_+ \cup \gamma_1^+ \cup \gamma_2^+$ has ω_{z_1} measure which is $o(1)$ as $\delta \to 0$, by the Beurling projection theorem. (In fact, $\omega_{z_1} = O(\sqrt{\delta})$). We construct γ_1^- and γ_2^- similarly at z_- and note that if ξ is not separated from z_0 by $\mathrm{cap}(z_1, \xi)$ then it is separated from z_1 by either $s^+ = L_+ \cup \gamma_1^+ \cup \gamma_2^+$ or $s^- = L_- \cup \gamma_1^- \cup \gamma_2^-$. Since the Green lines ending at ξ and $\xi_\pm$ form an angle of size α at z_1 and therefore sweep out harmonic measure $\frac{\alpha}{2\pi}$, this latter possibility cannot occur if δ is sufficiently small because the sets s^+ and s^- respectively separate $\xi_\pm$ from z_1. □

We need one more property of $\mathrm{cap}(z_1, \xi)$. Namely, that of all Brownian travelers starting from z_0 which hit $\mathrm{cap}(z_1, \xi)$ before hitting $\partial\Omega$, a large proportion of them hit $\mathrm{cap}(z_1, \xi)$ "near"' z_1. To prove it, we will apply lemmas 2.6 and 2.7 using harmonic measure and Green's function with pole at z_0 defined in the component Ω' of $\Omega \setminus \mathrm{cap}(z_1, \xi)$ which contains z_0. To make a precise statement, we will need to define, for a given constant $c > 0$ and a cap $\mathrm{cap}(z_1, \xi)$

$$T_c = \{z \in \mathrm{cap}(z_1, \xi) : d_\Omega(z) > c d_\Omega(z_1)\}.$$

We these notations we have

Lemma 3.2. *Given $\varepsilon > 0$ there is $c > 0$ not depending on Ω, such that*

$$\omega_{z_0}(T_c, \Omega') > (1 - \varepsilon)\omega_{z_0}(cap(z_1, \xi), \Omega')$$

Proof.

Consider the segment $L^+ \subset \mathrm{cap}(z_1, \xi)$ and mark off points $\{a_k\}$ on it so that $a_0 = z_+$ and for $k > 0$, a_k is the midpoint of a_{k-1} and ξ_+. Let $L_k^+ = [a_k, a_{k+1}]$ denote the segment in L^+ joining a_k and a_{k+1}. Let C_k denote the circle with diameter L_k^+ and let b_k denote the point of $C_k \cap \Omega$ on a radius of C_k perpendicular to L_k^+.

By lemma 2.8 we have ($\approx$ means "is comparable to")

$$\omega_{z_0}(L_k^+, \Omega') \approx g_{\Omega'}(z_0, b_k).$$

By lemma 2.6, lemma 2.7 and Harnack's inequality, there is $\delta > 0$ such that

$$\begin{aligned} g_{\Omega'}(z_0, b_k) &\approx g_{\Omega'}(z_0, b_1)(1 - \delta)^k \\ &\approx \omega_{z_0}(L_1^+, \Omega')(1 - \delta)^k \end{aligned}$$

where $g_{\Omega'}$ denotes the Green function for the domain Ω'. We repeat the construction on L^- (getting the same $\delta > 0$). Given $\varepsilon > 0$, after summing a geometric series, we see that for $K \geq \frac{\log \varepsilon\delta}{\log(1-\delta)}$ we have

$$\omega_{z_0}(\bigcup_{k \geq K}(L_k^+ \cup L_k^-), \Omega') \leq \varepsilon \omega_{z_0}(L_1^+ \cup L_1^-, \Omega')$$

so that

$$\omega_{z_0}(\ell^+ \cup \ell^- \cup \bigcup_{0 \leq k < K}(L_k^+ \cup L_k^-), \Omega') \geq (1 - \varepsilon)\omega_{z_0}(\mathrm{cap}(z_1, \xi), \Omega').$$

The set which is measured on the left hand side of this inequality is clearly contained in T_c for some $c > 0$ depending only on ε. □

Notice that if $0 \le u \le 1$ is harmonic in Ω, then by Harnack's inequality there is $c > 0$ depending on K (and therefore depending on ε) such that if $u(z_1) < 1-\delta$ then $u(z) < 1-c\delta$ for all $z \in \ell^+ \cup \ell^- \cup \bigcup\limits_{0\le k<K} (L_k^+ \cup L_k^-)$. It will be convenient below to refer to this set of z in the cap as the "top" part of the cap.

With these properties of the "caps", we are ready for the proof of Theorem 1.1. In the case of L^∞ boundary data f, it suffices by linearity and a limiting argument to consider $f = \chi_E$ (the indicator function) where E is a Borel subset of $\partial\Omega$. Lemma 2.5, Lemma 2.4 and Doob's version of Fatou's theorem together imply that $\limsup\limits_{z\to\xi, z\in\ell_\xi} \omega(z,E) = 1$ for ω_{z_0} almost every $\xi \in E$. By letting ε decrease to zero through a countable sequence, it suffices to show for a fixed $\varepsilon > 0$ that

$$\omega_{z_0}(\{\xi \in E : \liminf_{z\to\xi, z\in\ell_\xi} \omega(z,E) < 1-\varepsilon\}) = 0.$$

Denote the set whose measure is considered above by A_ε. For each $\xi \in A_\varepsilon$ we have a sequence $\{z_n\}$ such that $z_n \to \xi$ with $z_n \in \ell_\xi$ and $\omega(z_n,E) < (1-\varepsilon)$. We construct the caps $\mathrm{cap}(z_n,\xi)$. The caps correspond to intervals on $\{z : g(z_0,z) = \lambda\}$ for large $\lambda > 0$, where g denotes the Green function with pole at z_0. By the Vitali covering lemma we may choose a collection of them with disjoint bases which cover E. Consider the caps constructed this way as a first generation. We would like to construct successive nested generations so that a Brownian traveler starting on the top of an n^{th} generation cap has probability less than $1-c_1\varepsilon$ of hitting an $(n+1)^{\text{st}}$ generation cap. Here c_1 is a numerical constant which will become explicit below. Suppose for the moment that we can do this and let C_n denote the union of the n^{th} generation caps. Let Ω_n be the component of $\Omega \setminus C_n$ which contains z_0 and let $C_n = G_n \cup B_n$ where G_n is the union of the top parts of the caps and B_n is the union of the leftover parts. Let τ denote the first exit time from Ω for a Brownian motion starting from z_0 and let τ_n similarly denote the first exit time for Ω_n. By the strong Markov property for Brownian motion (used in the second line below) we then have

$$\begin{aligned}
\mathbb{P}^{z_0}\{X_\tau \in A_\varepsilon\} &\le \mathbb{P}^{z_0}\{X_{\tau_n} \in C_n\} \\
&= \int\limits_{C_{n-1}} \omega(z,C_n,\Omega_n)\,\omega(z_0,dz,\Omega_{n-1}) \\
&= \int\limits_{G_{n-1}} \omega(z,C_n,\Omega_n)\,\omega(z_0,dz,\Omega_{n-1}) + \int\limits_{B_{n-1}} \omega(z,C_n,\Omega_n)\,\omega(z_0,dz,\Omega_{n-1}) \\
&\le (1-c_1\varepsilon) \int\limits_{G_{n-1}} \omega(z_0,dz,\Omega_{n-1}) + \int\limits_{B_{n-1}} \omega(z_0,dz,\Omega_{n-1}) \\
&\le (1-c_1\varepsilon) \int\limits_{C_{n-1}} \omega(z_0,dz,\Omega_{n-1}) + c_1\varepsilon \int\limits_{B_{n-1}} \omega(z_0,dz,\Omega_{n-1}) \\
&\le (1-c_1\varepsilon) \int\limits_{C_{n-1}} \omega(z_0,dz,\Omega_{n-1}) + c_1\varepsilon^2 \int\limits_{C_{n-1}} \omega(z_0,dz,\Omega_{n-1})
\end{aligned}$$

$$
\begin{aligned}
&\le (1-c_2\varepsilon)\int\limits_{C_{n-1}} \omega(z_0,dz,\Omega_{n-1})\\
&= (1-c_2\varepsilon)\mathbb{P}^{z_0}\{X_{\tau_{n-1}}\in C_{n-1}\}.
\end{aligned}
$$

In passing from the fifth line to the sixth, we have used lemma 3.2. By iteration, it follows that $\mathbb{P}^{z_0}\{X_\tau \in A_\varepsilon\} = 0$.

That we can construct successive generations with the properties described above follows from Doob's version of Fatou's theorem by way of the fact that

$$\limsup_{z\to\xi, z\in\ell_\xi} \omega(z,E) = 1 \quad \text{a.e. } d\omega.$$

In each first generation cap we can construct an auxilliary second generation using the (stopping) condition $u > 1-\varepsilon'$ for some ε' which is sufficiently small compared to ε. If $\operatorname{cap}(z_1,\xi)$ is a first generation cap then the probability that a Brownian motion started from z_1 hits the auxilliary second generation caps before exiting Ω must be bounded above by $1-\varepsilon''$ for some $\varepsilon'' > 0$. Otherwise, estimating as above we find that $\omega(z_1, A_\varepsilon, \Omega) > 1-\varepsilon$ if ε' is sufficiently small. The harmonic measure of the base of $\operatorname{cap}(z_1,\xi)$ is as close as we like to $\frac{\alpha}{\pi}$ if δ is sufficiently small. So, by taking α sufficiently close to π and δ sufficiently small, we see that the harmonic measure ω_{z_1} of the auxilliary caps contained in $\operatorname{cap}(z_1,\xi)$ will be bounded above by $1-\varepsilon''/2$. Beneath each auxilliary cap, we now construct the "real" second generation caps using the condition $u < 1-\varepsilon$, and our second generation now has the required properties.

With more work, this argument can be extended to the case that the boundary function f is in L^p for $1 \le p \le \infty$ and it provides some insight into the structure of the exceptional sets in Fatou's theorem. We will, however, change tactics for the general case and consider the maximal function

$$u^*(\xi) = \sup_{z\to\xi, z\in\ell_\xi} |u(z)|, \qquad \xi \in \partial\Omega.$$

By the standard line of reasoning, in order to get our Fatou type convergence result, it suffices to prove a weak type inequality of the form

$$\omega_{z_0}(\{u^* > \lambda\}) \le \frac{C}{\lambda}\int\limits_{\partial\Omega} |f|\, d\omega_{z_0}. \tag{3.1}$$

For in the case that f is continuous, the required convergence follows from Wiener's theorem (or from the argument given above) and in the general case, if we are given $\varepsilon > 0$ and $f \in L^p$ we can write $f = g+h$ where g is continuous and h has $\|h\|_p < \varepsilon^2$. By linearity we obtain

$$\omega_{z_0}(\{\xi\in\partial\Omega : |\limsup_{z\to\xi, z\in\ell_\xi} u - \liminf_{z\to\xi, z\in\ell_\xi} u| > \varepsilon\}) \le \frac{C}{\varepsilon}\int\limits_{\partial\Omega} |h|\, d\omega_{z_0} < \varepsilon$$

which proves the theorem.

To get the weak type inequality (3.1), we note first that by linearity, and at the expense of the constant C, it suffices to consider the boundary function f to be non-negative. We construct, with the same notations for the caps as before, a first generation of caps using the condition $u > \lambda$. We define $\tilde{f}$ to be the boundary function on Ω_1 which is equal to u on C_1 and agrees with f on $\partial\Omega \cap \partial\Omega_1$. Using the properties of the caps established earlier, we then have

$$\begin{aligned}
\int_{\partial\Omega} f\, d\omega_{z_0} &= \int_{\partial\Omega_1} \tilde{f}\, \omega(z_0, d\xi, \Omega_1) \\
&\geq \int_{G_1} \tilde{f}\, \omega(z_0, d\xi, \Omega_1) \\
&\geq C\lambda\, \omega(z_0, G_1, \Omega_1) \\
&\geq C\lambda\, \omega_{z_0}(\{u^* > \lambda\})
\end{aligned}$$

which completes our proof.

4. Remarks and open questions

1. It has been observed here and in several places in the literature that there is a dictionary relating notions from potential theory and homogeneous Markov processes and that these two subjects provide equivalent tool kits for studying the same problems. See for example Øksendal (1983) and Sharpe (1986). Important results in analysis have been discovered first by probabilistic arguments and then have later been reproved by classical means. The maximal function characterization of H^p by Burkholder, Gundy and Silverstein (1971) is a prominent example of this.

There is a nice result about harmonic measure in $\mathbb{R}^3$ which has only been given a probabilistic proof. In Tsirelson (1997), Tsirelson showed

Theorem 4.1 (Tsirelson)**.** *If Ω_i is a domain in $\mathbb{R}^n$ for $i = 1, 2, 3$ with harmonic measure ω_i, there is no measure which is mutually absolutely continuous with respect to each of the measures ω_i.*

The theorem says that no matter how three (or more) domains may intertwine in some complicated way, their common boundary can only be given positive charge by at most two of the measures at a time. (Consider the example of a vascular system, a lymphatic system, and some third type of circulation system with the surrounding tissue as boundary.) The heart of the proof is the definition of an appropriate invariant quantity on filtrations which has different values for the Brownian Filtration and the filtration of the Walsh Brownian Motion. The paper Tsirelson (1997), which is full of ideas, contains a challenge to potential theorists to find a classical proof.

2. As was mentioned in section 1, it is not known whether domains $\Omega \subset \mathbb{R}^n$ have property $\mathscr{F}$. The techniques in this paper can be modified to give a proof of this for simply connected domains with some bounded geometry, such as the non-tangentially accessible (NTA) domains from Jerison and Kenig (1982, p.93). What should be true for all NTA domains and not just simply connected ones is that almost every Green line should eventually be a non-tangential approach curve. By the results of Jerison and Kenig (1982), this would obviate the previous statement for simply connected NTA domains. We hope to address this in a future paper.

3. In O'Neill (Preprint) we give a probabilistic proof of the McMillan Twist point theorem. The theorem, from McMillan (1969), states that if $\Omega \subset \mathbb{C}$ is simply connected then with respect to harmonic measure, almost every point $\xi \in \partial\Omega$ is either the vertex of a triangle contained in Ω or has the property that any curve Γ in Ω which ends at ξ has

$$\limsup_{w \to \xi, w \in \Gamma} \arg(w - \xi) = +\infty$$

$$\liminf_{w \to \xi, w \in \Gamma} \arg(w - \xi) = -\infty.$$

Here arg denotes a fixed single valued branch of $\arg(\cdot - \xi)$ in Ω. Fatou's theorem (Theorem 1.1) is required in the proof and so the proof here helps to keep the argument in O'Neill (Preprint) completely free of conformal mapping. The ideas in O'Neill (Preprint) may be extended to domains in $\mathbb{R}^n$ with controlled geometry (such as the NTA domains) and this will be addressed in a future paper.

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