# A PROBLEM OF MCMILLAN ON CONFORMAL MAPPINGS.

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Abstract. We answer one of two questions asked by McMillan in 1970 concerning distortion at the boundary by conformal mappings of the disk.

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## 1. Introduction

The purpose of this note is to answer a question of J.E McMillan concerning boundary behavior of conformal mappings which was raised in the paper [4]. In that paper, McMillan gave a sufficient geometric condition for a subset of the boundary of a domain to have harmonic measure zero and used it to prove a result which we will describe below. A similar geometric lemma was the key to the original proof of the twist point theorem in [5]. The reader can refer to both of McMillan's papers and to [6] for background on these problems and more generally to [1], [3] and [7] for the ideas used in this paper.

We will use $\omega(z_0, F, \Omega)$ to denote the harmonic measure of the set $F$ in the domain $\Omega$ from the point $z_0$. Let $\mathbb{D}$ denote the unit disk in the complex plane and let $f : \mathbb{D} \to \Omega$ be a conformal map. Let $A$ denote the set of all ideal accessible boundary points $f(e^{i\theta})$ of $\Omega$ when $f$ has the nontangential limit $f(e^{i\theta})$ at $e^{i\theta}$. Note that points of $A$ are prime ends of $\Omega$ so that a single complex coordinate may represent more than one point of $A$.

Let $D(a, r)$ denote a disk with center $a$ and radius $r$. Choose $r_0 < d(f(0), A)$ where $d$ denotes Euclidean distance. For each $a \in A$ and for each $r < r_0$ let $\gamma(a, r) \subset \partial D(a, r)$ be the crosscut of $\Omega$ seperating $a$ from $f(0)$ which can be joined to $a$ by a Jordan arc in $\Omega \cap D(a, r)$. Let $L(a, r)$ denote the Euclidean length of $\gamma(a, r)$ and let $U(a, r) = \bigcup_{r' < r} \gamma(a, r')$.

Let

$$A(a,r) = \int_0^r L(a,\rho)\,d\rho$$

denote the Lebesgue measure of $U(a,r)$.

McMillan proved

**Theorem 1.1.** *The set of $a \in A$ such that*

$$\limsup_{r\to 0} \frac{A(a,r)}{\pi r^2} < \frac{1}{2}$$

*has harmonic measure zero.*

Notice that this theorem implies that the set of $a \in A$ such that

$$\limsup_{r\to 0} \frac{L(a,r)}{2\pi r} < \frac{1}{2}$$

has harmonic measure zero.

McMillan also gave an example of a domain for which both

$$\limsup_{r\to 0} \frac{A(a,r)}{\pi r^2} = 1 \quad \omega \quad a.e.$$

and

$$\liminf_{r\to 0} \frac{A(a,r)}{\pi r^2} = 0 \quad \omega \quad a.e.$$

(implying the corresponding limits for $\frac{L(a,r)}{2\pi r}$) and conjectured that

$$E_1 = \{a \in A : \liminf_{r\to 0} \frac{A(a,r)}{\pi r^2} > \frac{1}{2}\}$$

and

$$E_2 = \{a \in A : \liminf_{r\to 0} \frac{L(a,r)}{2\pi r} > \frac{1}{2}\}$$

must be sets of harmonic measure zero.

Here, we will verify McMillan's conjecture that the set $E_2$ must always have zero harmonic measure.

## 2. There are no points of density in $f^{-1}(E_2)$.

With the notations and definitions of the introduction we prove

**Theorem 2.1.** *The harmonic measure of the set $E_2$ is zero.*

*Proof.* For any positive integers $m$ and $k$, let

$$E_{m,k} = \{a \in A |\, L(a,r) > (\frac{1}{2} + \frac{1}{m})2\pi r \quad \forall r < \frac{1}{k}\}.$$

Since $E_2$ is the countable union of sets $E_{m,k}$, it suffices to show that each $E_{m,k}$ has harmonic measure zero.

We will require the following lemma (see [7], pg 142) which is a consequence of results of Beurling, [2].

**Lemma 2.1.** *Let $f$ map $\mathbb{D}$ conformally into $\mathbb{C}$ and let $0 < \delta < 1$. If $z \in \mathbb{D}$ and $I$ is an arc of $\mathbb{T}$ with $\omega(z, I) \geq \alpha > 0$ then there exists a Borel set $B \subset I$ with $\omega(z, B) > (1-\delta)\omega(z, I)$ such that*

$$|f(\xi) - f(z)| \leq \Lambda(f(S)) < K(\delta, \alpha) d_f(z) \qquad \text{for } \xi \in B$$

*where $\Lambda$ denotes linear measure, $d_f(z)$ is the euclidean distance from $f(z)$ to the boundary of $f(\mathbb{D})$, $S$ is the non-euclidean segment from $z$ to $\xi$ and where $K(\delta, \alpha)$ depends only on $\delta$ and $\alpha$.*

The basic idea of the proof of theorem 2.1 is that since points of $E_{m,k}$ are separated from $f(0)$ by circular arcs of wide angle and large radius, if $f^{-1}(E_{m,k})$ has a point of density then lemma 2.1 will provide enough wide angled circular arcs of a fixed radius to wrap around on themselves and disconnect the domain $\Omega$.

Suppose then that $\eta \in \mathbb{T}$ is a point of density of $f^{-1}(E_{m,k})$ and let $I$ denote an arc of $\mathbb{T}$ centered at $\eta$.

Given $\delta_1 > 0$ we can choose $I$ such that

$$\frac{|f^{-1}(E_{m,k}) \cap I|}{|I|} > (1 - \delta_1). \tag{1}$$

Given $\delta_2 > 0$ we can find $0 < r(I, \delta_2) < 1$ such that

$$\omega((1 - r(I, \delta_2))\eta, I, \mathbb{D}) = 1 - \delta_2$$

and this determines the point $z_I = (1 - r(I, \delta_2))\eta$.

If we are given $\delta_3 > 0$ then if $\delta_1$ is sufficiently small, (1) implies that

$$\omega(f^{-1}(E_{m,k}), z_I, \mathbb{D}) > (1 - \delta_3).$$

By lemma 2.1, if we are given $\delta_4 > 0$ then there is a Borel set $B \subset I$ such that

$$\omega(z_I, B, \mathbb{D}) > (1 - \delta_4)(1 - \delta_2)$$

and such that

$$|f(\xi) - f(z_I)| < K(\delta_4, (1 - \delta_2))d_f(z_I) \quad \forall \xi \in B. \tag{2}$$

It follows that

$$\omega(f^{-1}(E_{m,k}) \cap B, z_I, \mathbb{D}) > 1 - (\delta_2 + \delta_3 + \delta_4 - \delta_2\delta_4) \tag{3}$$

and that (2) holds for all $\xi \in f^{-1}(E_{m,k}) \cap B$. Notice that the constant $K$ only depends on $\delta_2$ and $\delta_4$.

Since $f(\eta) \in A$ we can choose $I$ so that $Kd_f(z_I) << \frac{1}{k}$ where $k$ is the integer in the definition of $E_{m,k}$. The finite number of steps required to get a contradiction in the construction to follow will only depend on the number $m$ in the definition of $E_{m,k}$. By choosing a sufficiently small arc $I$, we can arrange that in each step of our construction, the positive number

$$\delta \equiv \delta_2 + \delta_3 + \delta_4 - \delta_2\delta_4$$

is small enough so that the construction can proceed to the next step. We assume that these conditions hold on the size of the interval $I$.

Let $w_0 = f(0)$, $w_1 = f(z_I)$, $d_0 = d_f(z_I)$ and let $x_1$ be a point of $\partial\Omega$ such that $|x_1 - w_1| = d_0$. Let the letters $c_1, c_2, \dots$ denote positive constants which will be assumed to be sufficiently small in each step of the construction but will ultimately depend only on the number $m$ in the definition of the set $E_{m,k}$ and not on $f$, $\Omega$, or $\delta$. Let $C_1, C_2, \dots$ denote other constants which may be purely numerical or which may depend only on the number $m$.

First let $0 < c_0 << 1$ and $c_1 << \frac{\pi}{m}c_0$. We will see that these choices allow for rotation by a fixed positive angle of certain separating circular arcs in consecutive steps of the construction to follow. The arc of $\partial D(x_1, c_1 d_0)$ which intersects the interior of $D(w_1, d_0)$ extends to a crosscut of $\Omega$ and determines a unique subdomain $U_1 \subset \Omega$ not containing $w_1$. We proceed to find a point close to $x_1$ which is contained in $E_{m,k}$. By Harnack's inequality,

$$\omega(w_1, \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0), \Omega) \geq C_1 \omega(w_1', \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0), \Omega)$$

where $w_1'$ is the point on the line between $w_1$ and $x_1$ such that $|x_1 - w_1'| = \frac{c_1 d_0}{2}$. By the comparison principle for harmonic measure and the Beurling projection theorem, ([1], pg.43),

$$\omega(w_1', \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0), \Omega) \geq C_2 > 0.$$

So by lemma 2.1 and equation (3), if $\delta$ is sufficiently small, ($\delta << C_1 C_2$), there is a constant $C_3$ such that

$$\omega(w_1, \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0) \cap E_{m,k}, \Omega) \geq C_3 > 0.$$

Choose a point $x_1^* \in \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0) \cap E_{m,k}$. If $c_0$ is sufficiently small then the arc of $\{z \in \mathbb{C} : |x_1^* - z| = c_0 d_0\}$ which intersects $D(w_1, d_0)$ has an angle greater than $\pi(1 - \frac{1}{2m})$. This arc must therefore be part of the crosscut whose length is $L(x_1^*, c_0 d_0) > \pi(1 + \frac{1}{m})$. Denote by $\overline{ab}$ the segment with endpoints $a \in \mathbb{C}$ and $b \in mathbbC$. Let $w_1^*$ be the point on $\overline{x_1^* w_1}$ with $|x_1 - w_1^*| = c_0 d_0$ and consider the annulus

$$R_1 = \{z \in \mathbb{C} : (1 - c_2)|x_1^* - w_1^*| < |x_1^* - z| < (1 + c_2)|x_1^* - w_1^*|\}$$

where $c_2 << \frac{\pi}{m} c_0$. Let $S_1$ be the component of $R_1 \cap \Omega$ which intersects $D(w_1, d_0)$ and let $x_2$ be a point of $\partial\Omega \cap S_1$ such that $\overline{x_1^* x_2}$ has minimal angle clockwise from $\overline{x_1^* w_1^*}$.

Let $S_1^*$ denote the sector of $R_1$ clockwise between $\overline{x_1^* w_1^*}$ and $\overline{x_1^* x_2}$. The circular arc $\partial D(x_2, c_2 d_0) \cap S_1^*$ is part of a crosscut of $\Omega$ which determines a unique subdomain $U_2$ of $\Omega$ not containing $w_1^*$. By an argument similar to the previous one using Harnack's inequality, the comparison principle for harmonic measure and the Beurling projection theorem but now in the annular sector $S_1$, it follows that

$$\omega(w_1, \partial\Omega \cap \partial U_2 \cap D(x_2, c_2 d_0), \Omega) > C_4 > 0$$

We remark that $C_4$ depends on $c_0, c_1, c_2$ and therefore only on $m$ and that the remaining constants $C_j$ may have similar dependence on $m$.

A simple geometric argument shows that there is a point $x_2^*$ in $D(x_2, c_2 d_0) \cap E_{m,k}$ and a constant $c_3 > 0$ determined by the diameter of the $E_{m,k} \cap D(x_2, c_2 d_0)$ such that the set of distances

$$\{|x_2^* - w| : w \in D(x_1^*, c_1 d_0) \cap \partial\Omega\}$$

contains an interval $J_1$ of length greater than $c_3 d_0$.

Let $R_2 = \{w \in \mathbb{C} : |w - x_2^*| \in J_1\}$ and let $S_2$ be the component of $R_2 \cap \Omega$ which intersects $S_1$. Each of the circular arcs of $S_2$ centered at $x_2^*$ is a crosscut of $\Omega$. If there is such a crosscut $L_1 \subset S_2 \cap \Omega$ which does not separate $x_2^*$ from $w_0$ then we repeat the above construction of $S_2$ but in the counterclockwise direction from $\overline{x_1^* w_1^*}$. Then any circular arc $L_2 \subset S_2 \cap \Omega$ centered at $x_2^*$ which intersects $S_1$, separates $x_2^*$ from $w_0$. For otherwise, $w_0$ is contained in both subdomains of $\Omega$ determined by the concave sides of $L_1$ and $L_2$. Since $w_0$ lies on the convex side of any circular arc which defines $L(a, r)$ for some $a \in A$ and $r > 0$ and therefore of any arc of $S_1$, this is impossible. If one choice of $S_2$, clockwise or counterclockwise from $\overline{x_1^* w_1^*}$, fails to separate $x_2^*$ from $w_0$ we choose the other. Otherwise, the construction can continue, as described below, in both directions until the non-separating case occurs and after that point, a topological argument similar to the above allows the construction to continue in the remaining direction.

We have now arranged that each of the circular arcs of $S_2$ centered at $x_2^*$ separates $x_2^*$ from $w_0$ and can be joined to $x_2^*$ by a Jordan arc lying inside $S_1$. Therefore, since $x_2^* \in E_{m,k}$, each circular arc of $S_2$ has an angular measure greater than $(1 + \frac{2}{m})\pi$. Let $w_2$ be a point of $S_2 \cap S_1$ and let $x_3$ be a point of $\partial\Omega \cap \overline{S_2}$ which minimizes the clockwise angle from $\overline{x_2^* w_2}$ to $\overline{x_2^* x_3}$. Let $S_2^*$ denote the sector of $R_2$ clockwise between $\overline{x_2^* w_2}$ and $\overline{x_2^* x_3}$. As before the circular arc $\partial D(x_3, c_3 d_0) \cap S_2^*$ extends to a crosscut of $\Omega$ which determines a unique subdomain of $U_3 \subset \Omega$ not containing $w_1$. The same harmonic measure argument as before but now done in the union of annular corridors $S_1 \cup S_2$ shows that

$$\omega(w_1, \partial\Omega \cap \partial U_3 \cap D(x_3, c_3 d_0), \Omega) > C_6 > 0.$$

If $\delta > 0$ is sufficiently small, then as before, lemma 2.1 and (3) imply that

$$\omega(w_1, \partial\Omega \cap \partial U_3 \cap D(x_3, c_3 d_0) \cap E_{m,k}, \Omega) > C_7 > 0$$

and we find $x_3^* \in \partial\Omega \cap \partial U_3 \cap D(x_3, c_3 d_0) \cap E_{m,k}$ such that the set of distances

$$\{|x_3^* - w| : w \in D(x_2^*, c_3 d_0) \cap \partial\Omega\}$$

contain an interval $J_3$ of length greater than $c_4 d_0$, where $c_4$ depends only on the previous $c_i$ and on $m$. Note that since the constants satisfy $c_i << c_0 \frac{\pi}{m}$, there is a numerical constant $c > 0$ such that the clockwise angle from $\overline{x_1^* x_2^*}$ to $\overline{x_2^* x_3^*}$ is at least $(1 + \frac{c}{m})\pi$. The construction continues in this way so that having found annular corridors $S_1, \dots S_j$ with centers $x_1^*, x_2^*, \dots x_j^*$ we find $x_{j+1}^* \in E_{m,k}$ so that there is an interval of distances $J_j$ between $x_{j+1}^*$ and the part of $\partial\Omega$ in a disk of radius $c_\ell d_0$ centered at $x_j^*$. The intersection of the annulus centered at $x_{j+1}^*$ determined by $J_j$ with $\Omega$ contains a component $S_{j+1}$ which intersects $S_j$. Concentric circular arcs of this annular piece separate $x_{j+1}^*$ from $w_0$ (or else the construction continues in the other direction) and each such circular arc can be joined to $x_{j+1}^*$ through the annular corridor $S_j$ by a Jordan arc contained in the circle. Therefore, each such arc has an angle greater than $(1 + \frac{2}{m})\pi$. Let $w_{j+1}$ be a point of $S_{j+1} \cap S_j$ and find $x_{j+2}$ which minimizes the clockwise angle between $\overline{x_{j+1}^* w_{j+1}}$ and $\overline{x_{j+1}^* x_{j+2}}$. The construction can continue if $\delta > 0$ is sufficiently small since the harmonic measure of the end of $S_{j+1}$ near $x_{j+2}$ from $w_1$ in $S_1 \cup S_2 \cup \dots S_{j+1} \cup D(w_1, d_0)$ is greater than some positive numerical constant.

But it is clear from the construction that the union of annular corridors $S_1 \cup \cdots \cup S_j$ must wrap around on itself after a finite number of steps which only depends on $m$. The union of annular corridors thus formed, being a subset of $\Omega$, would contain a closed curve in $\Omega$ whose interior component contains the points $x_i^* \in \partial\Omega$. Since $\Omega$ is simply connected, this contradiction shows that $f^{-1}(E_{m,k})$ does not contain a point of density and therefore must have measure zero. Therefore $E_{m,k}$ has harmonic measure zero in $\Omega$.

□

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