Math 152
Solutions for assignment 1.

#8 a) $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}\Big(\underbrace{\sqrt{1-\frac{n-1}{N-1}}}^{\text{neglect}}\Big)$

$$\simeq \sqrt{\frac{1/5 \cdot 4/5}{100}} = \frac{1}{25}$$

$$P(|\hat{p}-p| \geq \delta) = 1 - P(|\hat{p}-p| < \delta)$$

$$= 1 - P(-\delta < \hat{p}-p < \delta)$$

$$= 1 - P(-25\delta < 25(\hat{p}-p) < 25\delta)$$

$$\approx 1 - (2\Phi(25\delta) - 1) = 2(1-\Phi(25\delta))$$

$$= .025$$

$$1 - \Phi(25\delta) = .0125$$

$$\Phi(25\delta) = .9875.$$

$$\delta = \frac{\Phi^{-1}(.9875)}{25.} \approx \frac{2.24}{25} \approx .09$$

b) If $\hat{p} = .25$

$$s_{\hat{p}}^2 = \frac{\hat{p}(1-\hat{p})}{n-1}\Big(\overbrace{1-\frac{n}{N}}^{\text{neglect}}\Big) \simeq \frac{(.25)(.75)}{100} = \frac{3}{1600}$$

$$s_{\hat{p}} \approx \frac{\sqrt{3}}{40}$$

a 95% confidence interval is

$$\left(\hat{p} - \Phi^{-1}(.975)\, s_{\hat{p}} \,,\; \hat{p} + \Phi^{-1}(.975)\, s_{\hat{p}}\right)$$

$$\cong \left(.25 - 1.96\,\frac{\sqrt{3}}{40} \,,\; .25 + 1.96\,\frac{\sqrt{3}}{40}\right)$$

$$\simeq (.165, .335).$$

which contains $p = .2$

#28.

a) $P(Y) = P(Y/1)P(1) + P(Y/2)P(2)$

$$= pq + (1-p)(1-q)$$

$$= (2p-1)q + (1-p)$$

b) $q = \dfrac{r+p-1}{2p-1}$

c) $E(R) = r$ by Theorem A

Let $Q = \dfrac{R+p-1}{2p-1}$

then $E(Q) = \dfrac{1}{2p-1}\left(E(R) + p - 1\right) = \dfrac{r+p-1}{2p-1} = q$

d) $\mathrm{Var}(R) \approx \dfrac{r(1-r)}{n}$ by Thm B.

e) $$\mathrm{Var}(Q) = \mathrm{Var}\left(\frac{R}{2p-1} + \frac{p-1}{2p-1}\right)$$

$$= \left(\frac{1}{2p-1}\right)^2 \mathrm{Var}(R)$$

$$\approx \frac{1}{(2p-1)^2} \frac{r(1-r)}{n}.$$

34.

$p_1 \approx .03$
$p_2 \approx .40$

a) $$\sqrt{\frac{p_1(1-p_1)}{n}} \leq .01 \implies n \geq \frac{(.03)(.97)}{(.01)^2}$$

$$= 3(97) = 291$$

Since we only know an approximate size for p_1 this should be taken as a rough estimate of the necessary sample size.

34 cont.

$$\sqrt{\frac{p_2(1-p_2)}{n}} \le .01 \implies n \ge \frac{(.4)(.6)}{(.01)^2} = 2400.$$

With $n = 2400$ we have

$$\sqrt{\frac{(.03)(.97)}{2400}} \simeq .0035.$$

and $$\sqrt{\frac{(.4)(.6)}{2400}} \simeq .01$$

for the actual standard errors.

b) $$\sqrt{\frac{p_1(1-p_1)}{n}} \le \frac{.03}{10} = .003$$

$$\implies n \ge \frac{(.03)(.97)}{(.003)^2} \simeq 3234$$

$$\sqrt{\frac{p_2(1-p_2)}{n}} \le \frac{.40}{10} \implies n \ge \frac{(.4)(.6)}{(.04)^2} \simeq 150.$$

$n = 3234$ is roughly the required sample size.

```
#exercise 35 chapter 7
>
> t<-c(104,109,111,109,87,86,80,119,88,122,91,103,99,108,96,104,98,98,83,107,79,87,94,92,97)
>
> mean(t) # unbiased estimate of the population mean.
[1] 98.04
>
> var(t)
[1] 133.7067
>
> sum((t-mean(t))^2)/24 # this is the same as var(t) and is denoted by s^2 in the text.
[1] 133.7067
>
> var(t)* 1999/2000 # an unbiased estimate of the population variance as explained at the bot
tom of page 211.
[1] 133.6398
>
> (1/25)*var(t)*(1-(25/2000))# an unbiased estimate of the variance of the sample mean.
[1] 5.281413
>
> sqrt((1/25)*var(t)*(1-(25/2000)))
[1] 2.298133
>
> 98.04 -1.96*sqrt(5.28) # left hand endpoint of 95% C.I.
[1] 93.53626
>
> 98.04 +1.96*sqrt(5.28) #right hand endpoint of 95% C.I.
[1] 102.5437
>
```

Yes

28c)

No.

$$R = \frac{1}{n}\sum_{i=1}^{n} R_i$$

$$R_i = \begin{cases} 1 & \text{if } i\text{th sample member says Yes} \\ 0 & \text{if '' '' '' says no.} \end{cases}$$

$R =$ # of yes answers.

$$E(R) = \frac{1}{n}\sum_{i=1}^{n} E(R_i) = E(R_1).$$

$$E(R_1) = \text{Prob}\{R_1 = 1\}.$$

$$= pq + (1-p)(1-q).$$

$$= (2p-1)q + (1-p).$$

Math 152 Assignment 2 Solutions

```
> #Problem 3, chapter 8
>
> #The data:
>
> c1<-c(213,128,37,18,3,1,0,0,0,0,0,0,0)
>
> c2<-c(103,143,98,42,8,4,2,0,0,0,0,0,0)
>
> c3<-c(75,103,121,54,30,13,2,1,0,1,0,0,0)
>
> c4<-c(0,20,43,53,86,70,54,37,18,10,5,2,2)
>
> #Their respective sample means (which give the mle estimates for the parameter):
>
>  lambda1<-sum(c1*(0:12))/400
>
>  lambda2<-sum(c2*(0:12))/400
>
>  lambda3<-sum(c3*(0:12))/400
>
>  lambda4<-sum(c4*(0:12))/400
>
> lambda1
[1] 0.6825
>
> lambda2
[1] 1.3225
>
> lambda3
[1] 1.8
>
> lambda4
[1] 4.68
>
>
> # Standard errors for the sample means
>
>  s1<-sqrt(sum(c1*(0:12)))/400
>
>  s2<-sqrt(sum(c2*(0:12)))/400
>
>  s3<-sqrt(sum(c3*(0:12)))/400
>
>  s4<-sqrt(sum(c4*(0:12)))/400
>
> #95% confidence intervals for the estimated parameters
>
> c(lambda1-1.96*s1,lambda1 + 1.96*s1)
[1] 0.6015387 0.7634613
>
> c(lambda2-1.96*s2,lambda2 + 1.96*s2)
[1] 1.2098 1.4352
>
> c(lambda3-1.96*s3,lambda3 + 1.96*s3)
[1] 1.668519 1.931481
>
> c(lambda4-1.96*s4,lambda4 + 1.96*s4)
[1] 4.467994 4.892006
>
> #Comparison of the data with expected values from the mle distribution:
> #At this point, the comparison is merely by "common sense".
> # The fits look resonably good, but there could be some question about the fit for data set
 3.
>
> round(((lambda1^(0:12))/(factorial(0:12)))*exp(-lambda1) *400)
 [1] 202 138  47  11   2   0   0   0   0   0   0   0   0
>
```

```
> c1
 [1] 213 128  37  18   3   1   0   0   0   0   0   0   0
>
> round(((lambda2^(0:12))/(factorial(0:12)))*exp(-lambda2) *400)
 [1] 107 141  93  41  14   4   1   0   0   0   0   0   0
>
> c2
 [1] 103 143  98  42   8   4   2   0   0   0   0   0   0
>
> round(((lambda3^(0:12))/(factorial(0:12)))*exp(-lambda3) *400)
 [1]  66 119 107  64  29  10   3   1   0   0   0   0   0
>
> c3
 [1]  75 103 121  54  30  13   2   1   0   1   0   0   0
>
> round(((lambda4^(0:12))/(factorial(0:12)))*exp(-lambda4) *400)
 [1]  4 17 41 63 74 69 54 36 21 11  5  2  1
>
> c4
 [1]  0 20 43 53 86 70 54 37 18 10  5  2  2
>
```

8. Exercise 7 is solved in 152notes0012. We refer to the results from it.

a) $\hat{p} = \frac{1}{\bar{X}}$ for both m.l.e. and method of moments.

From the data:

$$\bar{X} = \frac{48\cdot1 + 31\cdot2 + 20\cdot3 + 9\cdot4 + 6\cdot5 + 5\cdot6 + 4\cdot7 + 2\cdot8 + 1\cdot9 + 1\cdot10 + 2\cdot11 + 1\cdot12}{48+31+20+9+6+5+4+2+1+1+2+1}$$

$\cong 2.79$, So $\hat{p} \simeq .358$

b) The asymptotic variance of the m.l.e is

$$\frac{p^2(1-p)}{n} \approx \frac{\hat{p}^2(1-\hat{p})}{130.} \approx .00063$$

So the approx. standard error is $\approx .025$.

Our 95% confidence interval for p is

$$(.358 - 1.96(.025), .358 + 1.96(.025))$$

$$\cong (.309, .407).$$

c) Expected counts with this value of $\hat{p}$ are

$$130(1-\hat{p})^{(k-1)}\cdot(\hat{p}) \qquad k = 1, \ldots, 12$$

where k is the number of hops

c) Continued

rounding to the nearest integer, this is

k	observed	expected.
1	48	47
2	31	30
3	20	19
4	9	12
5	6	8
6	5	5
7	4	3
8	2	2
9	1	1
10	1	1
11	2	1
12	1	0

So the fit looks good.

d). With a uniform prior, the posterior distribution is (see #7 part d)

(beta) $\beta(n+1, n(\bar{X}-1)+1) \doteq \beta(131, 233.7)$

The posterior mean is $\frac{n+1}{n\bar{X}+2} \cong .359$

and the posterior standard deviation is (see appendix A)

$\sqrt{\frac{131 \cdot (233.7)}{(131+233.7)^2(131+233.7+1)}} \simeq .025$

#32 Calculations attached

a) $\hat{\mu} = \bar{X}$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

b). $$\left(\bar{X} - \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2) \, , \; \bar{X} + \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2) \right)$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

$\alpha = .10 \, , \; .05 \, , \; .01.$

$n = 16.$

$$\left(\frac{n\hat{\sigma}^2}{\chi^2_{n-1}(\alpha/2)} \, , \; \frac{n\hat{\sigma}^2}{\chi^2_{n-1}(1-\alpha/2)} \right)$$

$n = 16 \, , \; \alpha = .10 \, , \; .05 \, , \; .01$

c) $a \le \hat{\sigma}^2 \le b$ iff $\sqrt{a} \le \hat{\sigma} \le \sqrt{b}$

or by bootstrapping.

d) about 4 times larger

since $S^2 \to \sigma^2$ and $t_{n-1}(\alpha/2) \to z(\alpha/2)$

as $n \to \infty$.

$z(\alpha/2) = \Phi^{-1}(1-\alpha/2)$ where Φ is the c.d.f of $N(0,1)$

```
> #Problem 32, chapter 8
>
> #The data:
>
> data<-c(5.3299,4.2537,3.1502,3.7032,1.6070,6.3923,3.1181,6.5941,3.5281,4.7433,.1077,1.5977,
5.4920,1.7220,4.1547,2.2799)
>
> data
 [1] 5.3299 4.2537 3.1502 3.7032 1.6070 6.3923 3.1181 6.5941 3.5281 4.7433 0.1077 1.5977
[13] 5.4920 1.7220 4.1547 2.2799
>
> #mle estimates for the mean and variance for normally distributed data:
>
> mean(data)
[1] 3.610869
>
> mean((data-mean(data))^2)
[1] 3.204461
>
> # Confidence intervals for the mean:
>
> s<-sqrt((16/15)*mean((data-mean(data))^2))
>
> s
[1] 1.848808
>
> #90%
> c(mean(data) -(s/4)*qt(.95,15),mean(data) +(s/4)*qt(.95,15))
[1] 2.800605 4.421132
>
> #95%
> c(mean(data) -(s/4)*qt(.975,15),mean(data) +(s/4)*qt(.975,15))
[1] 2.625708 4.596029
>
> #99%
> c(mean(data) -(s/4)*qt(.995,15),mean(data) +(s/4)*qt(.995,15))
[1] 2.248892 4.972846
>
>
>
> #Confidence intervals for the variance:
>
> #90%
>
> c(16*mean((data-mean(data))^2)/qchisq(.95,15),16*mean((data-mean(data))^2)/qchisq(.05,15))
[1] 2.051201 7.061256
>
>
> #95%
> c(16*mean((data-mean(data))^2)/qchisq(.975,15),16*mean((data-mean(data))^2)/qchisq(.025,15)
)
[1] 1.865201 8.187521
>
> #99%
> c(16*mean((data-mean(data))^2)/qchisq(.995,15),16*mean((data-mean(data))^2)/qchisq(.005,15)
)
[1]  1.563089 11.143735
>
> #Confidence intervals for the standard deviation:
>
> #90%
> sqrt(c(16*mean((data-mean(data))^2)/qchisq(.95,15),16*mean((data-mean(data))^2)/qchisq(.05,
15)))
[1] 1.432201 2.657302
>
> #95%
> sqrt(c(16*mean((data-mean(data))^2)/qchisq(.975,15),16*mean((data-mean(data))^2)/qchisq(.02
```

```
5,15)))
[1] 1.365724 2.861384
>
> #99%
>
> sqrt(c(16*mean((data-mean(data))^2)/qchisq(.995,15),16*mean((data-mean(data))^2)/qchisq(.00
5,15)))
[1] 1.250236 3.338223
>
> #Now bootstrap for a 95% confidence interval for the standard deviation.
> #Since the square root of an mle is not necessarily an mle,
> #the asymptotic methods do not apply here.
>
> #The estimated value:
>
> sqrt(mean((data-mean(data))^2))
[1] 1.790101
>
>
>
> f<-function(x){ d<-rnorm(16,mean=mean(data),sd=sqrt(mean((data-mean(data))^2)))
+
+                 sqrt(mean((d-mean(d))^2))}
>
> boot<-sapply(1:1000,f)
>
> o<-order(boot)
>
> deltalow<-boot[o][25]- sqrt(mean((data-mean(data))^2))
>
> deltahigh<-boot[o][975]- sqrt(mean((data-mean(data))^2))
>
> c(sqrt(mean((data-mean(data))^2))-deltahigh,sqrt(mean((data-mean(data))^2))-deltalow)
[1] 1.20522 2.44710
>
>
>
```

```
> #Exercise 46
>
> #Read in and attach the data:
>
> #whales<-read.table("C:/Documents and Settings/Mike/Desktop/Math 152/whales.txt",header=T)
>
> #whales<-read.table("U:/Math 152/whales.txt",header=T)
>
> whales<-read.table("http://math.cmc.edu/moneill/Math152/Handouts/whales.txt",header=T)
>
>
> attach(whales)
>
> names(whales)
[1] "times"
>
> #a)
>
> hist(times)
> #attached: figure 46.1
>
> hist(times,breaks=30)
> #attached: figure 46.2
>
> #A gamma distribution is a plausible guess for a model since the histogram is skewed.
>
>
> #b)
> #Method of moments estimates:
>
> varhat<-mean(times^2)-mean(times)^2
>
> lambdahatm<-mean(times)/varhat
>
> lambdahatm
[1] 1.318771
>
>
>
>
> alphahatm<-lambdahatm* mean(times)
>
> alphahatm
[1] 0.799174
>
> #Compare the fitted distribution with the normalized histogram
>
>
> f<-function(x){((lambdahatm^alphahatm)/gamma(alphahatm))*x^(alphahatm-1)*exp(-lambdahatm*x)
}
>
> hist(times,probability=T)
>
> curve(f(x),add=T)
>
> #attached:figure 46.3
>
> hist(times,breaks=30,prob=T)
>
> curve(f(x),add=T)
>
> #attached:figure 46.4
>
>
> #c)
>
> #Maximum likelihood estimates:
```

```
>
> a<-(sum(log(times))-210*log(mean(times)))/210
>
> f<-function(x) {log(x)+a-digamma(x)}
>
> uniroot(f,interval=c(0.01,3),lower=0.01,upper=2)
$root
[1] 1.595414

$f.root
[1] -1.328309e-06

$iter
[1] 8

$estim.prec
[1] 6.103516e-05

>
> uniroot(f,interval=c(0.01,3),lower=0.01,upper=2)[1]
$root
[1] 1.595414

>
> alphahatmle<-as.numeric(uniroot(f,interval=c(0.01,3),lower=0.01,upper=2)[1])
>
> alphahatmle
[1] 1.595414
>
>
> lambdahatmle<-alphahatmle/mean(times)
>
> lambdahatmle
[1] 2.632701
>
>
> #d) The mle estimate gives a more effective summary of the data since the method of moments
 estimate
> #is too large for small values of the argument. (Though the fit is not impressively good fo
r either estimate)
>
> g<-function(x,alphahat,lambdahat){((lambdahat^alphahat)/gamma(alphahat))*x^(alphahat-1)*exp
(-lambdahat*x)}
>
> curve(g(x,alphahatmle,lambdahatmle),add=T)
>
> attached: figure 46.5
Error: syntax error
>
> hist(times,probability=T)
> curve(g(x,alphahatmle,lambdahatmle),add=T)
>
> #attached: figure 46.6
>
> #e)
> #>bootstrapping method of moments
>
> resamp<-function(x){rgd<-rgamma(210,shape=alphahatm,rate=lambdahatm)
+                     mean(rgd^2)-mean(rgd)^2->var
+                     lambdastar<-mean(rgd)/var
+                     alphastar<-lambdastar*mean(rgd)
+                     lambdastar
+  }
>
> hist(sapply(1:1000,resamp))
> #not included
>
```

```
> data<-sapply(1:1000,resamp)
> slambdahat<-sqrt((1/1000)*sum((data-mean(data))^2))
> slambdahat
[1] 0.2245093
>
>
>
> resamp<-function(x){rgd<-rgamma(210,shape=alphahatm,rate=lambdahatm)
+                    mean(rgd^2)-mean(rgd)^2->var
+                    lambdastar<-mean(rgd)/var
+                    alphastar<-lambdastar*mean(rgd)
+                    alphastar
+  }
>
> hist(sapply(1:1000,resamp))
> #not included
>
> data<-sapply(1:1000,resamp)
> salphahat<-sqrt((1/1000)*sum((data-mean(data))^2))
> salphahat
[1] 0.1153530
>
>
>
> #f)The mle estimates have a higher variability than the method of moments estimates (shown
below).
> #But, as mentioned above, the mle estimate is a better summary of the data.
>
> resamp<-function(s){rgd<-rgamma(210,shape=alphahatmle,rate=lambdahatmle)
+
+               f<-function(x) {210*log(x)-210*log(mean(rgd)) + sum(log(rgd)) -210*digamma(x)}
+
+                      alphahat<-uniroot(f,interval=c(0.01,3),lower=0.01,upper=3)[1]
+
+                      alphahat<-as.numeric(alphahat)
+
+                     lambdahat<-alphahat/mean(rgd)
+
+                     alphahat
+
+                     #lambdahat
+  }
>
> hist(sapply(1:1000,resamp))
> #not included
>
> alphadata<-sapply(1:1000,resamp)
> salphahat<-sqrt((1/1000)*sum((alphadata-mean(alphadata))^2))
> salphahat
[1] 0.1448516
>
>
>
>
> resamp<-function(s){rgd<-rgamma(210,shape=alphahatmle,rate=lambdahatmle)
+
+               f<-function(x) {210*log(x)-210*log(mean(rgd)) + sum(log(rgd)) -210*digamma(x)}
+
+                      alphahat<-uniroot(f,interval=c(0.01,3),lower=0.01,upper=3)[1]
+
+                      alphahat<-as.numeric(alphahat)
+
+                     lambdahat<-alphahat/mean(rgd)
+
+                     #alphahat
+
+                     lambdahat
```

```
+  }
>
> hist(sapply(1:1000,resamp))
>
> lambdadata<-sapply(1:1000,resamp)
> slambdahat<-sqrt((1/1000)*sum((lambdadata-mean(lambdadata))^2))
> slambdahat
[1] 0.2856847
>
>
> #g)
>
> #Bootstrapping for the mle  95% confidence intervals
>
>
> #For lambda:
>
> o<-order(lambdadata)
>
>
> deltalow<-lambdadata[o][25]- lambdahatmle
>
> deltahigh<-lambdadata[o][975]-lambdahatmle
>
> c(lambdahatmle-deltahigh,lambdahatmle-deltalow)
[1] 1.986277 3.102179
>
>
> #For alpha:
>
> o<-order(alphadata)
>
>
> deltalow<-alphadata[o][25]- alphahatmle
>
> deltahigh<-alphadata[o][975]-alphahatmle
>
> c(alphahatmle-deltahigh,alphahatmle-deltalow)
[1] 1.273099 1.837277
>
```

fig. 46.1

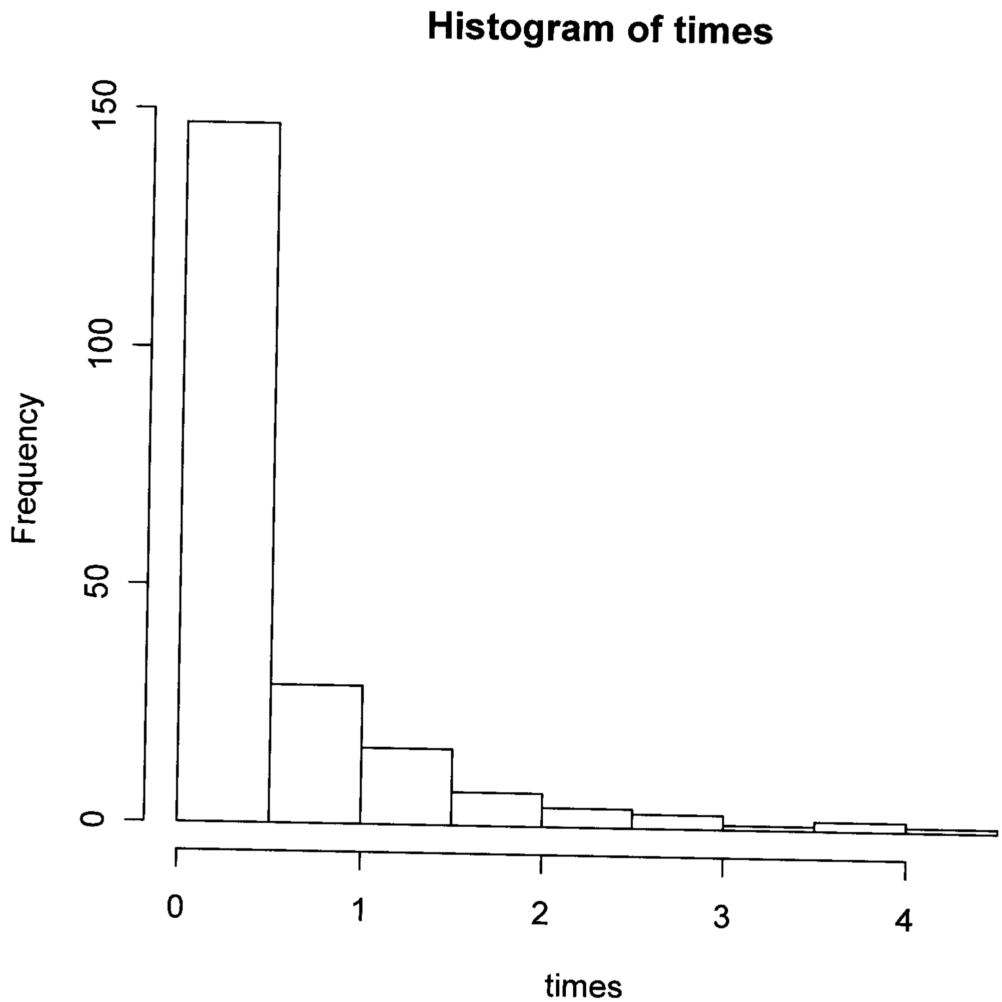


fig 46.2

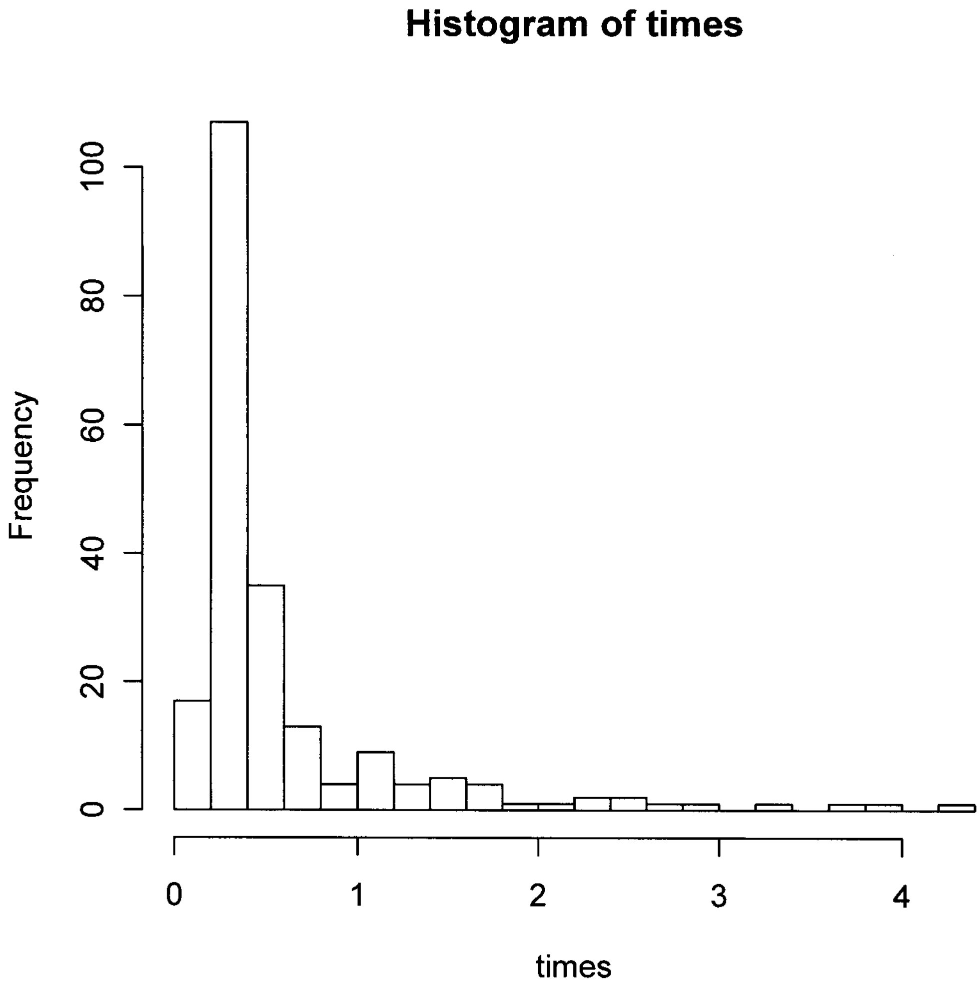


fig 46.3 Moments

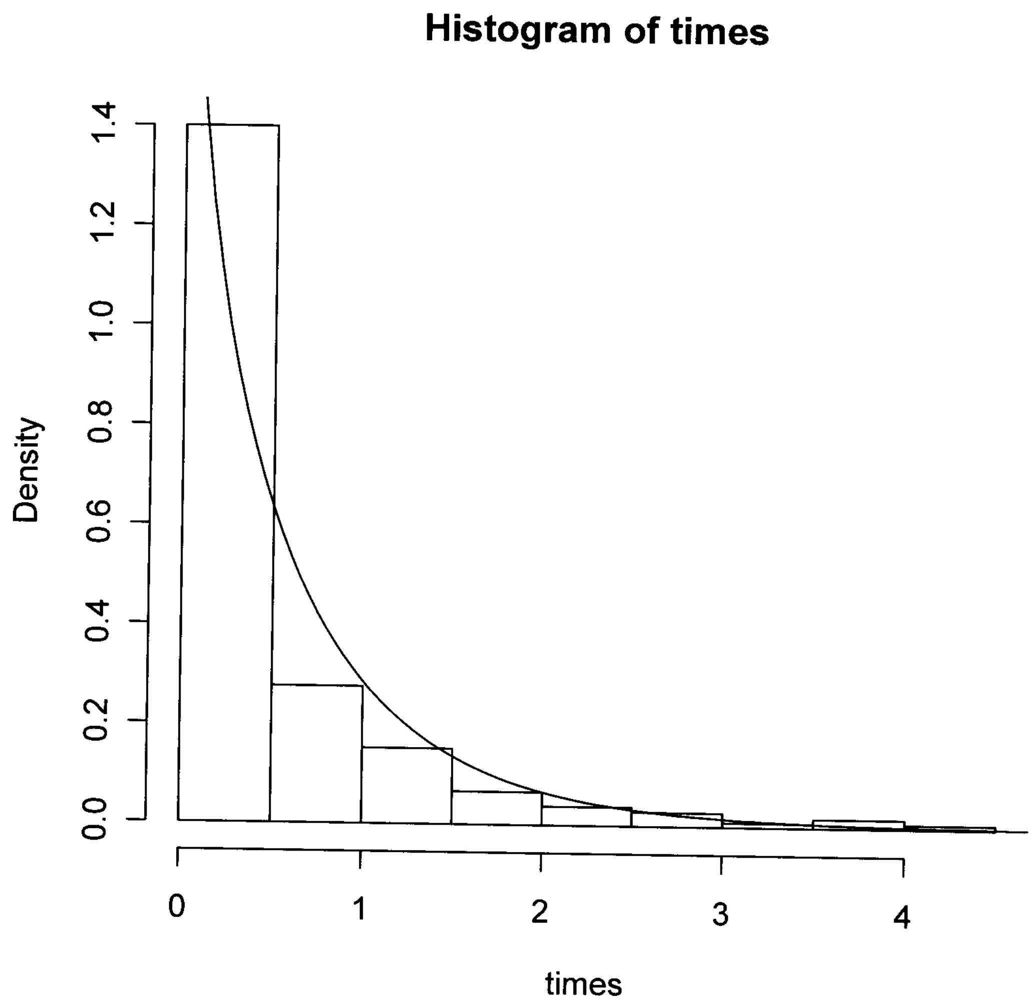


fig 46.4 Moments.

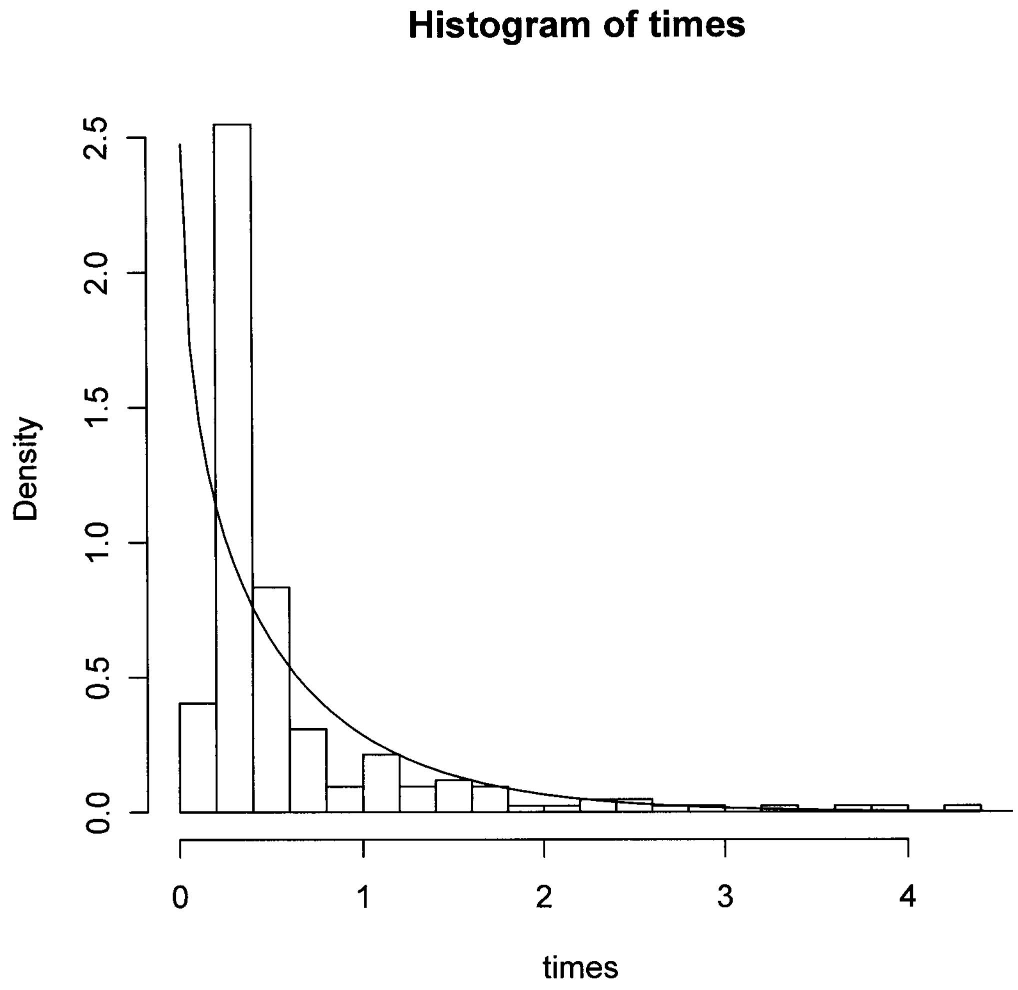


fig 46.5

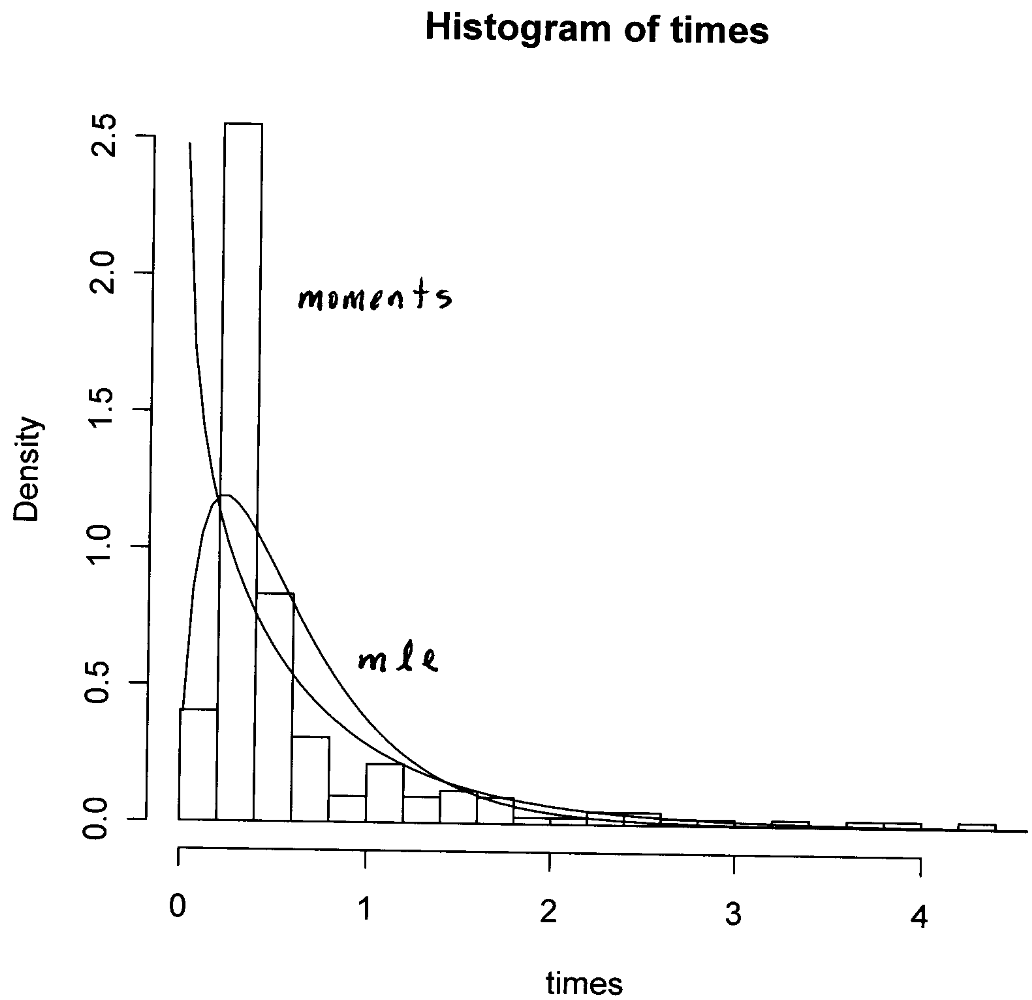
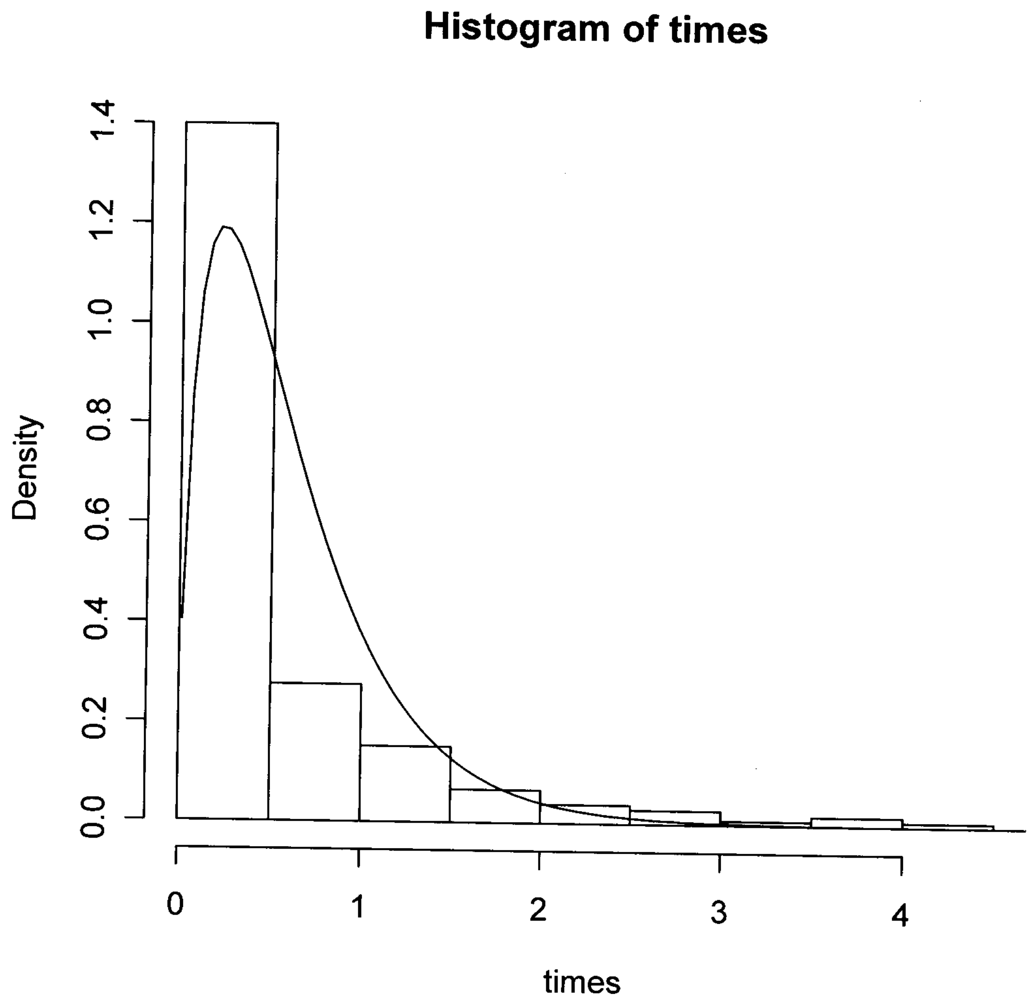

Histogram of times
moments
mle
Density
0.0
0.5
1.0
1.5
2.0
2.5
0
1
2
3
4
times

fig 46.6. MLE



68. a)

$$P(X_1 = x_1, \ldots, X_m = x_m \mid T = t)$$

$$= \frac{P(T=t \mid X_1 = x_1, \ldots X_m = x_m) \cdot P(X_1 = x_1, \ldots X_m = x_m)}{P(T=t)}$$

$$= 0 \quad \text{if} \quad x_1 + \cdots + x_m \neq t$$

and else,

$$= \frac{1 \cdot \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}}{\frac{(n\lambda)^{\Sigma x_i}}{(\sum_{i=1}^{n} x_i!)} e^{-n\lambda}} = \left(\frac{1}{n^t}\right) \frac{t!}{x_1! \cdots x_n!}$$

which is independent of λ.

Since this is just the multinomial coefficient, we could have guessed it without using Bayes rule.

Once the total # of events $T=t$ is given, the events are equally likely to accumulate for each of the variables.

b). $f_{\vec{X}}(x_1, \ldots, x_n \mid X_1 = a)$

$= 0$ if $x_1 \neq a$.

else $= \prod_{j=2}^{n} \frac{\lambda^{x_j} e^{-\lambda}}{x_j!}$

which is not independent of λ.

So X_1 is not sufficient

c) $P(X_1 = x_1, \ldots X_m = x_m)$

$= \prod_{j=1}^{m} \frac{\lambda^{x_j} e^{-\lambda}}{x_j!} = \frac{e^{-\lambda m} \lambda^{(\sum_{j=1}^{n} x_j)}}{\prod_{j=1}^{m} x_j!}$

$= g(T, \lambda) h(\vec{X})$

$T = \sum_{j=1} x_j$, $g(t, \lambda) = e^{-\lambda m} \lambda^t$

$h(\vec{x}) = \frac{1}{\prod_{j=1}^{m} x_j!}$